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Abstract: We consider here the problem of extrema for the Kac functional with long range, ferromagnetic interaction, and vorticity conditions at infinity which make it not weakly closed. Using a gradient-flow dynamics, we investigate local minima, showing strong analogies with the Ginzburg-Landau functional in infinite volume.

Keywords: Kac functional, Vortex, XY model, gradient flow dynamics, renormalization.

0. INTRODUCTION.

We consider here the XY or "planar rotator" model and its corresponding continuous version, with internal continuous symmetry group $O^+(2)$ and long range, ferromagnetic interaction. Such interactions were introduced by Lebowitz and Penrose in Statistical Physics as a generalization of the celebrated Kac, Uhlenbeck and Hemmer model, accounting for liquid vapour phase transitions, and giving mathematical foundations to Van der Waals theory.

A theorem of Dobrushin & Shlosman [Si,p.78] asserts that at any inverse temperature β, every infinite volume Gibbs state corresponding to a fairly general, translation invariant hamiltonian H with internal continuous symmetry group $O^+(2)$ on the 2-d lattice, is invariant (up to conjugation of charge) under the group $O^+(2)$. This is absence of breakdown of continuous symmetry. Later Bricmont, Fontaine and Landau discovered that this Gibbs state is unique. It presents many interesting, and yet not fully understood features, such as a particular form of phase transition at low temperature, which is characterized by the change of behavior in the correlation functions. For the XY system they were described by Kosterlitz & Thouless in term of topological vortices.

Here we look instead at the free energy functional, which provides, through the mean field approximation, a good approximation to the canonical Gibbs measure on the set of magnetizations m, i.e. suitables averages of spins over mesoscopic regions. Moreover, vortices can be enhanced by imposing some boundary condition on H at infinity.

One usually consider the vector spin hamiltonian on $\mathbf{Z}^2$ as a limit of hamiltonians (the thermodynamical limit) on finite lattices $\Lambda \to \mathbf{Z}^2$, with boundary conditions on Λ^c. Then the free energy functional takes the form

$$\begin{aligned}\mathcal{F}(m|m^c) &= \frac{1}{4}\int_\Lambda dr \int_\Lambda dr' J(r-r')|m(r)-m(r')|^2 \\ &+ \frac{1}{2}\int_\Lambda dr \int_{\Lambda^c} dr' J(r-r')|m(r)-m(r')|^2 + \int_\Lambda dr\big(f_\beta(m(r)) - f_\beta(m_\beta)\big)\end{aligned} \tag{0.1}$$

where $f_\beta(m) = -\frac{1}{2}|m|^2 + \frac{1}{\beta}I(m)$ is the free energy for the mean field approximation, and $I(m)$ denotes entropy function. Actually formula (0.1) holds also (formally) in the continuous case, see Appendix C. Systems of spins valued in $\{+1,-1\}$, i.e. the scalar case, with long range interaction are well understood, due in particular to a series of papers by Cassandro, DeMasi, Presutti and their collaborators, who studied in great detail interfaces and equilibrium shapes.

In an attempt to generalize this theory to the XY model [El-BoRo], the authors have looked for local minima of the free energy functional, and observed in numerical simulations, among other things, vortex configurations on finite lattices Λ, induced by the vorticity on the boundary Λ^c, which are very similar to those arising in solutions of Ginzburg-Landau equations, describing a superfluid or a supraconductor subject to a magnetic field. Related examples include the 't Hooft-Polyakov monopole and the Skyrme model.

We will consider in this paper a continuous version of the rotator, that complies to the methods of Functional Analysis. The minimization problem for the free energy functional (0.1) and its continuous version in infinite volume has the $O^+(2)$ symmetry, which we break by imposing on the magnetization m a vorticity condition at infinity of the form $m(x) = m_\beta e^{in\theta}$, $n \in \mathbf{N}\setminus 0$. Here $m_\beta \in]0,1[$ is the critical value for the mean field free energy, and we work at a sub-critical temperature, i.e. $\beta > 2$.

The obvious conjecture about a minimizer is that there is a radially symmetric vortex of degree n, expressed in polar coordinates as $m(x) = u_n(r)e^{in\theta}$, for some nonnegative function $u_n(r)$ with $u_n(r) \to m_\beta$ as $r \to \infty$, and $u_n(0) = 0$. For $n = 1$, this is called a *hedgehog*.

A similar conjecture holds in the case of Ginzburg-Landau equation in a disc, where it is known that the hedgehog ψ_1, uniquely defined, is stable, i.e. all the eigenvalues of the self-adjoint second variation operator L_{ψ_1},

$$(v|L_{\psi_1}v) = \frac{1}{2}\frac{d^2}{d\varepsilon^2}\mathcal{E}\big(\psi_1 + \varepsilon v\big)|_{\varepsilon=0} \tag{0.2}$$

are positive. This was obtained independently by Lieb and Loss [LiLo], and Mironescu [M].

In infinite volume, the situation is more subtle, due to the translational symmetry of the problem. In fact, Ovchinnikov and Sigal showed (still for Ginzburg-Landau equation) that for any n, there is a unique radially symmetric vortex ψ_n of degree n, that minimizes $\mathcal{E}_{\text{ren}}(\psi)$ among all functions of the form $\psi(x) = u_n(r)e^{in\theta}$. Here $\mathcal{E}_{\text{ren}}$ is a renormalized energy functional. The self-adjoint second variation operator L_{ψ_n} is decomposed as channel operators in Fourier modes ; the n-th mode is a positive operator, and has not 0 for an eigenvalue, but 0 is an eigenvalue for the $n+1$-th mode (due to the translational symmetry), and there are also negative eigenvalues for some higher modes. Thus we cannot conclude in this case to linear stability of the radially symmetric vortex.

Minimizers of the free energy of the scalar Kac model, and their stability were investigated by DeMasi, Presutti *et al.* In the 1-d case, they proved existence of a solution, called *instanton*, unique modulo translations, of the minimization problem subject to the condition that the magnetizations $m(x)$ tend to $\pm m_\beta$ as x tend to infinity. Their result was rederived lateron by Alberti and Bellettini [AlBe], and extended to higher dimensions, for a class of functions "varying only in a direction e". In fact, both functionals (Kac and Ginzburg-Landau) are not convex, and the free energy of Kac model is not even local, so that direct methods [Da] don't apply here.

Rearrangements methods were used in both [LiLo] for Ginzburg-Landau equation in a disc, and [AlBe] for the scalar Kac model. Actually, only partial convexity is achieved when restricting to the class of radially symmetric functions for Ginzburg-Landau, while for the scalar Kac model, convexity is retrieved, by rearrangements, on a set of increasing functions. But it is hard to figure out at least what *rearrangements* would mean in the vector (or complex) spin model. We follow here an alternative route, elaborated in the papers [AlBeCasPr], [DeM], [DeMOrPrTr], and culminating in [Pr], which consists in looking instead for extrema, verifying Euler-Lagrange equation, as limiting orbits for a certain dynamics, known in that context as the *gradient flow dynamics* T_t (see (1.11)). Free energy is a Lyapunov function for the gradient flow dynamics, and thus we can resort on methods used in parabolic equations.

Our main results are related to existence of radially symmetric solutions $m_n(x)$ for Euler-Lagrange equations $F_1(m)=0$ (see (1.8)) associated to a suitable renormalized free energy functional in infinite volume $\mathcal{F}_{\rm ren}$ defined in Proposition b.5. We have :

Theorem 0.1: Assume that J is on negative definite in the sense of [FrTo], and the convolution operator expressed in the $\langle e_n\rangle$-sector enjoys asymptotic properties described in Definition 1.1. Let $m(x)=(m_\beta+v(r))e^{in\theta}$, with $v\in\widetilde{X}_2^0$ (see (1.28)) such that $\mathcal{F}_{\rm ren}(m)<\infty$. Then any limit point m^* of $T_t m$, $t\to\infty$ (in the sense of uniform convergence on the compact sets) satisfies $F_1(m^*)=0$ and $\mathcal{F}_{\rm ren}(m^*)\leq\mathcal{F}_{\rm ren}(m)$.

Thus we can describe some local minima in the space of all configurations m. Next we study the spectrum of the n-th Fourier mode of the second variation operator L_{m^*} around m^*. Here we assume a particular form for J.

Theorem 0.2: Assume J is a Gaussian, and moreover that the critical point m^* verifies

$m^*(x) = (m_\beta + v(r))e^{in\theta}$, with $v^* \in \widetilde{X}_2^0$. Then the n-th Fourier mode of the second variation operator L_{m^*} (due to breaking the gauge invariance) has purely continuous spectrum.

(The decay property on v^* could certainly be removed, see Sect.3.) So we recover the result of [OvSi] for the Ginzburg-Landau functional relative to the n-th mode only. Continuous,, but also negative spectrum of L_{m^*} near 0 suggests that only "linear instability" (not exponential) may occur, in this mode, around the equilibrium. Nevertheless our conclusion remains much weaker than for the Ginzburg-Landau functional [OvSi], since we have no information about higher modes. Naive intuition suggests that 0 is an eigenvalue for the $n+1$-th Fourier mode, corresponding to the breaking of the translation group. So the "hedgehog conjecture" described above in the case of Ginzburg-Landau equation, remains largely open in case of the non local Kac functional (here the case $n = 1$ doesn't seem to play any special rôle).

1. The gradient-flow dynamics.

We look for a solution of the free energy variational problem, among all configurations with given degree $n \geq 1$, obtained as a limiting orbit of the gradient-flow dynamics. The minimization problem in infinite volume involves an approximation process in finite boxes Λ, and renormalization of the free energy. Definitions of the thermodynamic functions are given in Appendix C, and we refer to [El-BoRo] for details.

a) The infinite volume gradient flow dynamics.

We start to construct the gradient flow dynamics in $\mathbf{R}^2$ subject to a vorticity condition at infinity. It is convenient to express everything in Fourier modes, using the identification $L^2(\mathbf{R}^2) \approx \oplus_{j\in\mathbf{Z}} e_j L^2(\mathbf{R}^+; rdr)$, $e_j(\theta) = e^{ij\theta}$. In Appendix B we discuss some properties of the degree, and show that higher harmonics can be involved in the Fourier expansion of m, provided they decay sufficiently fast as $r \to \infty$; but because of non linearity, we choose a single component $j = n$, the $\langle e_n \rangle$ sector.

For given n, Euler-Lagrange equation for $\mathcal{F}_{\text{ren}}$ given by (b.28) is the same as we would obtain formally, i.e. without renormalizing the energy. It follows that it is of the form $F_1(m) = 0$, where

$$F_1(m) = -J * m + \frac{1}{\beta}\widehat{I}'(|m|)\frac{m}{|m|} \tag{1.8}$$

(recall the notation $\widehat{I}(|m|) = I(m)$ whenever I is rotation invariant.) Eqn.(1.8) is equivalent to $F_0(m) = 0$, where

$$F_0(m) = -m + f(\beta|J * m|)\frac{J * m}{|J * m|} \tag{1.9}$$

Here $f = (\widehat{I}')^{-1} = I_1/I_0$, and I_ν denotes the modified Bessel function of order ν. Solutions of (1.8) or (1.9) tend to cluster near the manifold $|m| = m_\beta$, which is the critical set for the free energy of mean field. Here $m_\beta > 0$ for $\beta > 2$ satisfies $m_\beta = f(\beta m_\beta)$.

Let also $\mathcal{F}u(\xi) = \int_{\mathbf{R}^2} e^{-ix\xi} u(x)dx$ and $H_n u(\rho) = \int_0^\infty J_n(r\rho)u(r)rdr$ denote Fourier and Hankel transformation, respectively. H_n is defined on the core $C_0^\infty(\mathbf{R}^+)$ of smooth, compactly supported functions on the half line. For $u \in C_0^\infty(\mathbf{R}^+)$, we have $H_n u(\rho) = \mathcal{O}(\rho^n)$ as $\rho \to 0$, and $H_n u(\rho) = \mathcal{O}(\rho^{-\infty})$ as $\rho \to \infty$. It extends [Ti] to a unitary operator on $L^2(\mathbf{R}; rdr)$, and $H_n^* = H_n$.

The convolution operator thus becomes $J* = \big(H_n \mathcal{F}\widehat{J} H_n\big)_{n\in\mathbf{Z}}$. We look for a radially symmetric solution of (1.9), of the form $m(x) = u(r)e^{in\theta}$, with $u(r) \to m_\beta$ as $r \to \infty$, and $u(0) = 0$. So we get, formally, $-A_n u + \frac{1}{\beta}\widehat{I}'(u) = 0$, $A_n = H_n \mathcal{F}\widehat{J} H_n$, or

$$-u + f(\beta A_n u) = 0 \tag{1.10}$$

Unless $\beta \leq 2$, in which case $u = 0$ is the unique solution of (1.10), and $m = 0$ the unique minimizer of $\mathcal{F}_{\text{ren}}$ (see [El-BoRo,Proposition 3.5] for the case of a lattice,) the map $u \mapsto f(\beta A_n u)$ is not a contraction, so following [Pr], we are led to consider the associated gradient-flow dynamics, and solve the "heat equation"

$$\frac{du}{dt} = -u + f(\beta A_n u), \ u(0, r) = u_0 \tag{1.11}$$

Although A_n is a bounded operator on $L^2(\mathbf{R}^+; rdr)$, we are faced with the problem that $u \notin L^2(\mathbf{R}^+; rdr)$ because of the condition $u(r) \to m_\beta$ as $r \to \infty$. We need the property, that A_n acts naturally upon functions having asymptotics near infinity. It should also map the constant function 1 to itself modulo L^2. So we introduce the :

Definition 1.1: We say that A_n has the asymptotic property iff, given χ a smooth cut-off equal to 1 near infinity, for all $k \in \frac{1}{2}\mathbf{Z}$, $A_n(\cdot)^{-k}\chi$ is a smooth function on the half-line, with $A_n((\cdot)^{-k}\chi)(r) \sim r^{-k}(1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots)$, in the sense of asymptotic sums, as $r \to \infty$. Moreover, if $a_1 = 0$ we say that A_n has the asymptotic property with vanishing subprincipal symbol.

We don't know if Definition 1.1 could actually be derived from similar arguments in Hankel's theorem leading to the inversion of H_n (see [Wa,p.458].) Operator A_n with the asymptotic property reminds us of a Pseudo-Differential Operator, with r as the fiber variable, and "hidden" phase coordinates that show up when writing the integral representation of Bessel functions. It is non-local (not only because convolution is not local, but more seriously because a change to polar coordinates always destroys locality,) with a priori only weak decoupling from 0 to ∞, see Appendix A.

Since f is only defined on $\mathbf{R}^+$, operator A_n must be also positivity preserving, i.e. $A_n u \geq 0$ almost everywhere (a.e.) if $u \geq 0$.

Example 1: J is a gaussian normalized in L^1, so that $\mathcal{F}\widehat{J}(\rho) = \exp[-p\rho^2]$, $p > 0$; we know [Wa,p.395] that $A_n(r, r')$ is given by Weber second exponential integral

$$\int_0^\infty \exp[-p\rho^2] J_n(r\rho) J_n(r'\rho)\rho d\rho = \frac{1}{2p}\exp[-(r^2 + r'^2)/4p] I_n(rr'/2p) \geq 0 \tag{1.12}$$

with equality only if $rr' = 0$. So A_n is clearly positivity preserving (more precisely, positivity improving, i.e. $A_n u > 0$ a.e. if $u \geq 0$, not identically 0,) and stationary phase arguments, together with the asymptotic expansion of modified Bessel function

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}}\big(1 - (n^2 - 1/4)/(2x) + \cdots\big), \quad x \to \infty \tag{1.13}$$

show that $r^k A_n((\cdot)^{-k}\chi)(r) \sim 1 + (k^2 - n^2)/r^2 + \cdots$, so A_n has the asymptotic property with vanishing subprincipal symbol.

Example 2: $\mathcal{F}\widehat{J}(\rho) = e^{-p\rho}(1 + p\rho)$, and $A_n(r, r')$ can be computed taking derivatives with respect to p, from the expression [Wa,p.389]

$$\int_0^\infty \frac{\exp[-p\rho]}{\rho} J_n(r\rho) J_n(r'\rho)\rho d\rho = \frac{1}{\pi\sqrt{rr'}} Q_{n-1/2}\big(\frac{r^2 + r'^2 + p^2}{2rr'}\big) \tag{1.14}$$

where $Q_{n-1/2}$ is a Legendre function of second kind, for which we have the integral representation $Q_{n-1/2}(\cosh\eta) = \int_\eta^\infty (2\cosh s - 2\cosh\eta)^{-1/2} e^{-ns} ds$, see [Le,p.174]. It is easy to check again that $A_n(r, r') \geq 0$.

Note that these J's are non negative definite in the sense of [FrTo], i.e. both $\widehat{J}$ and $\mathcal{F}\widehat{J}$ are ≥ 0, this excludes antiferromagnetic potentials, or potentials with non definite sign. Of course, $\mathcal{F}\widehat{J} \geq 0$ implies that A_n is a positive operator in the mean, i.e $(A_n v|v) \geq 0$ for all $v \in L^2(\mathbf{R}^+, rdr)$, but this is not sufficient for our purposes. The additional condition $(\mathcal{F}\widehat{J})'(0) = 0$ seems to ensure that A_n has the asymptotic property with vanishing subprincipal symbol. These features are suggested again by formal stationary phase expansions in the integral representation of Bessel functions, but from a ill-behaved phase that becomes rapidly oscillating for large r.

Our first result deals with existence and regularity of solutions of (1.10) in L^2 or in Sobolev spaces. This will not play the most important rôle in the sequel, but this is a very natural property. We introduce the closed convex set $\widetilde{W} = \{v \in L^2(\mathbf{R}^+, rdr) : v(r) + m_\beta \geq 0 \text{ a.e.}\}$ and $\widetilde{W}([0, T]) = C^0([0, T], \widetilde{W})$ with norm

$$\|v(t, r); \widetilde{W}([0, T])\| = \sup_{t\in[0,T]} \|v(t, r)\|_{L^2} \tag{1.18}$$

Note first that, since $\widehat{J} \geq 0$ and $\int J = 1$, we have $\mathcal{F}\widehat{J}(\rho) \leq 1$ and $\|A_n\| \leq 1$, where $\|\cdot\|$ denotes the (C^*-) norm of operators.

Theorem 1.2: Let as above A_n be positivity preserving, and enjoy the asymptotic property of Definition 1.1 with vanishing subprincipal symbol. Then for all $T > 0$ and all $v_0 \in \widetilde{W}$, there is a unique $v \in \widetilde{W}([0,T])$ such that $u = v + m_\beta$ verifies (1.11) with $u|_{t=0} = v_0 + m_\beta$.

Proof: Since A_n is positivity preserving, $A_n u \geq 0$ a.e. if $u \geq 0$ a.e. so (1.11) makes sense for $u = v + m_\beta$, $v \in \widetilde{W}$. Moreover, $A_n u = A_n v + v_1 + m_\beta$, $v_1 = m_\beta(A_n 1 - 1) \in L^2(\mathbf{R}^+)$ because of the asymptotic property with vanishing subprincipal symbol, and of Proposition a.1. Equation (1.11) with initial condition $v(0,r) = v_0(r)$ can be written in the integrated form

$$v(t,r) = e^{-t}v_0(r) + \int_0^t dt_1 e^{t_1 - t}[f(\beta A_n v(t_1, r) + \beta v_1(r) + \beta m_\beta) - f(\beta m_\beta)] \tag{1.19}$$

Let again $w(t,r) = v(t,r) - e^{-t}v_0(r)$, this rewrites as $w(t,r) = \Phi(t,r,w)$, with

$$\Phi(t,r,w) = \int_0^t dt_1 e^{t_1 - t}[f\big(\beta A_n w(t_1,r) + \beta m_\beta + \beta v_1(r) + \beta e^{-t_1} A_n v_0(r)\big) - f(\beta m_\beta)] \tag{1.20}$$

Given $v_0 \in \widetilde{W}$, denote by $\widetilde{W}_{v_0}([0,T])$ the closed convex set of $C^0([0,T], L^2)$ consisting of all functions $w(t,r)$ such that $w(t,r) + e^{-t}v_0(r) \in C^0([0,T], \widetilde{W})$. Because $f \geq 0$, Φ maps $\widetilde{W}_{v_0}([0,T])$ into itself. Moreover, since $\|f'\|_\infty = 1/2$, the estimate

$$|\Phi(t,r,w_1) - \Phi(t,r,w_2)| \leq \frac{\beta}{2} \int_0^t dt_1 e^{t_1 - t} |A_n(w_1 - w_2)(t_1, r)| \tag{1.21}$$

and the fact that $\|A_n\| \leq 1$ on $\widetilde{W}$ prove that if $\beta^2 T(1 - e^{-2T}) < 8$, then Φ is a contraction on $\widetilde{W}_{v_0}([0,T])$. Theorem 1.2 then follows from the group property. ♣.

Consider next the derivatives of u. Applying the radial field $r\partial_r$ to $J * m(x) = e^{in\theta} A_n u(r)$, $x = x_1 + ix_2 = re^{i\theta}$ we find, in the distributional sense :

$$e^{in\theta} r \frac{\partial}{\partial r} A_n u(r) = x_1 \frac{\partial J}{\partial x_1} * m(x) + x_2 \frac{\partial J}{\partial x_2} * m(x) \tag{1.22}$$

which shows, since $u = |m|$:

$$|\partial_r A_n u(t,r)| \leq \|\nabla J\|_1 \|u\|_\infty \tag{1.23}$$

$\|\cdot\|_1$ being the L^1 norm. On the other hand, using (c.6) we compute

$$\Big(\frac{\partial}{\partial r} + \frac{n}{r}\Big) A_n = B_n, \quad B_n(r,r') = \int_0^\infty d\rho \rho J_{n-1}(r\rho) J_n(r'\rho) \mathcal{F}\widehat{\mathcal{J}}(\rho) \tag{1.24}$$

where as before $B_n(r,r')$ stands for the kernel of operator B_n with measure rdr. We have

Lemma 1.3: With notations above, $\|B_n\| \leq 1$, $\|\frac{1}{r}A_n\| \leq C$, for some $C > 0$.

Proof: The first inequality results from $B_n^* B_n = H_n(\mathcal{F}\widehat{\mathcal{J}})^2 H_n$ and the fact that H_n is unitary on $L^2(\mathbf{R}^+, rdr)$. We use also (a.14) to write

$$\frac{1}{r}A_n = \frac{1}{2n}(H_{n-1}\big(\rho\mathcal{F}\widehat{\mathcal{J}}(\rho)\big)H_n + H_{n+1}\big(\rho\mathcal{F}\widehat{\mathcal{J}}(\rho)\big)H_n)$$

Since the multiplication by $\rho\mathcal{F}\widehat{\mathcal{J}}(\rho)$ is bounded on L^2 we get the second inequality. ♣

From this we can extend Theorem 1.2 to show regularity of the r-derivative u' of u. Namely, let $\widetilde{W}^1$ be the closed convex set $\{v \in \widetilde{W} : v(r) + m_\beta \leq 1 \text{ a.e., } v' \in L^2(\mathbf{R}^+, rdr)\}$, and $\widetilde{W}^1([0,T]) = C^0([0,T]; \widetilde{W}^1)$ with Sobolev norm as in (1.18). We have :

Proposition 1.4: Let $v \in C^0(\mathbf{R}^+, \widetilde{W})$ be the solution of (1.11) constructed in Theorem 1.2, with initial value $v_0 \in \widetilde{W}^1$. Then for all $T > 0$, $v \in \widetilde{W}^1([0,T])$.

Proof: With notations as in the proof of Theorem 1.2, we have

$$\begin{aligned} \frac{\partial w}{\partial r}(t,r) = \beta \int_0^t ds e^{s-t} f'\big(\beta A_n w(s,r) + \beta m_\beta + \beta v_1(r) + \beta e^{-s} A_n v_0(r)\big) \\ \times \frac{\partial}{\partial r}\big(A_n w(s,r) + v_1(r) + e^{-s} A_n v_0(r)\big) \end{aligned} \tag{1.25}$$

We denote by $\Phi'(t,r,w)$ the RHS of (1.25) and prove that Φ' is a contraction on $\widetilde{W}^1([0,T])$ if $T > 0$ is small enough. But this results also from Lemma 1.3, and the fact that $w \in \widetilde{W}([0,T])$. ♣

This Proposition extends easily by induction to all derivatives, so Sobolev embedding theorem shows that $r \mapsto v(t,r)$ inherits the regularity of its initial datum.

Existence and uniqueness result in L^2 however, falls far short of our needs to ensure existence of a limiting orbit satisfying (a.10) as $t \to \infty$, or to provide suitable asymptotics, essentially because we lack of a uniform bound on $\|v(t,r); \widetilde{W}([0,T])\|$ as $T \to \infty$. We restrict henceforth to continuous functions that tend to 0 sufficiently fast as $r \to \infty$. Such initial conditions will be used in the Barrier Lemma below. It turns out that the evolution equation doesn't either provide a uniform bound on the L^∞ norm of $u(t,r)$ but we can still obtain indirectly such estimates. Asymptotic property of operators A_n will be used as a hint to model our functional spaces. So we consider the space X_k^0 of functions $v \in C_0(\mathbf{R}^+)$, such that $\sup_{r \in [1,+\infty[} |r^k v(r)| < \infty$. It is easy to see that X_k^0 is a separable Banach space, with norm

$$\|v; X_k^0\| = \sup_{r \in [0,1]} |v(r)| + \sup_{r \in [1,+\infty[} |r^k v(r)| \tag{1.28}$$

For $k=2$, we have $X_k^0 \subset L^2$. In Appendix A, we give continuity properties of A_n acting on these spaces. Let also $\widetilde{X}_k^0$ be the closed convex subset of X_k^0 consisting of functions $v \in C_0(\mathbf{R}^+)$, such that $v(r)+m_\beta \geq 0$, and define $\widetilde{X}_k^0([0,T]) = C^0([0,T], \widetilde{X}_k^0)$.

Theorem 1.5: With the assumptions above, and if in addition A_n has vanishing principal symbol, then for all $T>0$ small enough, and all $v_0 \in \widetilde{X}_k^0$, $k=1,2$, there is a unique $v \in \widetilde{X}_k^0([0,T])$ such that $u = v + m_\beta$ verifies (1.11) with $u|_{t=0} = v_0 + m_\beta$.

Proof: Following the proof of Theorem 1.2, we need to check that Φ is a contraction on $X_{k,v_0}^0([0,T])$, the closed convex set of $C^0([0,T]; C_0(\mathbf{R}^+))$ consisting of functions $w(t,r)$ such that $w(t,r) + e^{-t}v_0(r) \in C^0([0,T], \widetilde{X}_k^0)$. First we estimate $|\Phi(t,r,w(r))|$, and write

$$|\Phi(t,r,w)| \leq \frac{\beta}{2}\int_0^t dt_1 e^{t_1-t}\big(|A_n w(t_1,r)| + |v_1(r)| + e^{-t_1}|A_n v_0(r)|\big) \tag{1.29}$$

By Proposition a.2, A_n is a bounded operator on X_k^0, and since $v_1 \in X_k^0$ by assumption, we have $w \in \widetilde{X}_k^0$; by the same remark as in the proof of Theorem 1.2, we conclude that $\Phi(t,r,w) \in \widetilde{X}_{k,v_0}^0$. Then (1.23) shows that Φ is a contraction on $\widetilde{X}_{k,v_0}^0([0,T])$, when $T>0$ is small enough, and the proof goes as in Theorem 1.2. ♣

Again, by the group property, we find $v \in C^0(\mathbf{R}^+, \widetilde{X}_k^0)$. From Theorem 1.5 we can infer existence of limit points of the orbits :

Corollary 1.6: With the same hypotheses as in Theorem 1.5, from any sequence $t_n \to +\infty$, we can extract a subsequence t_{n_j} such that $v(t_{n_j}, r) \to v^* \in C^0(\mathbf{R}^+)$ as $j \to \infty$ for the convergence on compact sets.

Proof: The family $u(t,r) = m_\beta + v(t,r)$ is clearly bounded by 1, since $m(t,x) = e^{in\theta}u(t,r)$, $x = re^{i\theta}$ solves the evolution equation corresponding to (1.10). Consider next $w(t,r)$ as in the proof of Proposition 1.4, (1.23) shows that $\frac{\partial}{\partial r}\big(A_n(w(s,r) + v_1(r) + e^{-s}A_n v_0(r)\big)$ is bounded uniformly in (t,r), so by integration of (1.25), $\partial_r w(t,r)$ is uniformly bounded, and so $|\partial_r v(t,r)| \leq C$. It follows that the family $v(t,r)$ is also equicontinuous, and the conclusion follows from Ascoli-Arzelà theorem. ♣

Let us extend once more our previous considerations. We enrich the structure of our Banach space X_k^0 by requiring some asymptotic behavior near ∞. Consider indeed the set Y_1^0 of functions $u \in X_1^0$, such that $ru(r)$ has a limit as $r \to \infty$, and if $\ell(u) = \lim_{r\to\infty} ru(r)$, the function $r(ru(r) - \ell(u))$ is bounded. It is easy to see that Y_1^0 is a separable Banach space, with norm

$$\|u; Y_1^0\| = \sup_{r\in[0,1[}|u(r)| + \sup_{r\in[1,+\infty[}|ru(r)| + \sup_{r\in[1,+\infty[}\big|r\big(ru(r) - \ell(u)\big)\big| \tag{1.30}$$

Similarly, consider the set Y_2^0 of functions $v \in X_2^0$, such that $rv(r)$ has a limit $\ell_0(v)$ as $r \to \infty$, $r^2v(r) - r\ell_0(v)$ has a limit $\ell_1(v)$ as $r \to \infty$, and the function $r^{1/2}(r^2v(r) - r\ell_0(v) - \ell_1(v))$ is bounded. It is easy to see that Y_2^0 is also a separable Banach space, with norm

(1.31)

$$\|v; Y_2^0\| = \sup_{r\in[0,1[} |v(r)| + \sup_{r\in[1,+\infty[} |r^2v(r)| + \sup_{r\in[1,+\infty[} |r^{1/2}\big(r^2u(r) - r\ell_0(v) - \ell_1(v)\big)|$$

By $\widetilde{Y}_k^0$, $k = 1, 2$ we denote also as before the closed convex subspace of Y_k^0 consisting of functions v such that $v(r) + m_\beta \geq 0$, and $\widetilde{Y}_k^0([0,T]) = C^0([0,T], \widetilde{Y}_k^0)$.

Proposition 1.7: With the assumptions of Theorem 1.5 then for all $T > 0$ small enough, and all $v_0 \in \widetilde{Y}_k^0$, $k = 1, 2$, there is a unique $v \in \widetilde{Y}_k^0([0,T])$ such that $u = v + m_\beta$ verifies (1.11) with $u|_{t=0} = v_0 + m_\beta$.

The proof goes along the same steps as this of Theorem 1.5, but this time we need also Proposition a.3 to control the last term in (1.30) or (1.31) for $\|v(t,\cdot); Y_k^0\|$. Note that we do not expect asymptotics beyond this order, on account of our accuracy in estimating A_n.

Whatever the class $\widetilde{X}_k^0$ or $\widetilde{Y}_k^0$ to which the initial datum does belong, we cannot ensure that v^* itself belongs to this set, nor even to $C_0(\mathbf{R}^+)$. Fortunately the main properties we shall use don't hinge upon v^* itself (except for Theorem 0.2, where we are lead to make an hypothesis about the short range of v^*,) but are rather inherited from those of the finite time evolution $v(t,r)$, $t > 0$. For this reason we shall also call a limiting orbit of the flow a X-*ghost*, stressing that it proceeds from an initial datum in X. Actually the phase picture for the whole dynamics may look quite complicated, unless we could prove uniqueness of the limiting orbits of (1.11), at least for a given u_0.

Now we extend the gradient-flow dynamics to those positive functions, which are merely continuous on $\mathbf{R}^+$ and bounded by 1. Indeed we shall eventually take a $\widetilde{X}_k^0$-ghost u^* as a new initial datum, but use $u^*(t,r)$ only with r in a compact set. We have :

Proposition 1.8: Let A_n as above be positivity preserving, but not necessarily with the asymptotic property. Then for all $T > 0$ small enough, and all $u_0 \in C^0(\mathbf{R}^+)$, $\|u_0\|_\infty \leq 1$, there is a unique $u \in C^0(\mathbf{R}^+)([0,T])$ such that $u(t,r)$ verifies (1.11) with $u|_{t=0} = u_0$.

The proof is omitted, for it goes as in Theorem 1.5, without the additional requirements on asymptotics at infinity. Iterates of $\Phi(t,r,w)$ to the fixed point converge uniformly for r on every compact of $\mathbf{R}^+$. Again, by the group property, we find $u \in C^0(\mathbf{R}^+; C^0(\mathbf{R}^+))$.

Next we consider various comparison theorems. As in the scalar case [Pr], the comparison theorem or maximum principle shows very useful in our situation. To fix the ideas, we state it for the full dynamics. Recall that $u^+ = v^+ + m_\beta, v^+ \in C^0(\mathbf{R}^+, \widetilde{X}_k^0)$ is a supersolution of the Cauchy problem (1.11) with initial datum $u_0^+ \in \widetilde{X}_k^0$ if $\|u^+(t,\cdot)\|_\infty \leq 1$, $u_0^+ \geq u_0$ and verifies

$$\frac{du^+}{dt} \geq -u^+ + f(\beta A_n u^+)$$

We define analogously a subsolution u^-. Because A_n is positivity improving, for all $t \geq 0$, $u^-(t,r) \leq u(t,r) \leq u^+(t,r)$. The first consequence of the maximum principle is that u doesn't increase beyond m_β if this holds for u_0. More precisely, we can show the following :

Proposition 1.9: Let $u(t,r) = v(t,r) + m_\beta$, $v \in C^0(\mathbf{R}^+, \widetilde{X}_k^0)$ be the solution of (1.11) with initial datum $u_0(r)$ satisfying $|u_0(r)| \leq \mu \leq 1$ for some $\mu \geq m_\beta$. Then $|u(t,r)| \leq \mu$ and for all $t \geq 0$.

The easiest way of proving this is to consider $m(t,r) = e^{in\theta}u(t,r)$, and the argument goes as in as in [El-BRo, Proposition 3.3]. This holds equally for the partial dynamics. Another consequence of the maximum principle is monotonicity :

Proposition 1.10: Let A_n be positivity preserving. If the initial datum $u_0(r)$ in (1.11) is an increasing function of r, then the same holds of $u(t,r)$ for all $t \geq 0$. Assume moreover A_n be positivity improving. If $u_0(r)$ is strictly increasing, then the same holds of $u(t,r)$ for all $t \geq 0$.

Proof: Let $a > 0$, and consider the new dynamics on $r \in]a, +\infty[$ given by $\frac{du_a}{dt} = -u_a + f(\beta A_n u_a)$, $u_a(0,r) = u_a$, where $u_a(r) = u_0(r-a)$ is the translate of u_0. Since $u_a(r) \leq u_0(r)$, we see that u is a subsolution of $\frac{du}{dt} = -u + f(\beta A_n u)$, so $u(t, r-a) \leq u(t,r)$ for all $t > 0$. ♣.

Note this implies the former result if we take $\lambda = m_\beta$ in Proposition 1.9. Of course these properties extend, by continuity, to the limit points $u^*(r)$.

When considering the spectral problem in Sect.3, we shall need to know that the limiting orbits belong also to $C^1(\mathbf{R}^+)$. Taking the r-derivative of u in (1.11) involves the derivative of $A_n u$ which we can compute using (1.24), but it is very hard to give estimates on B_n along the lines of Appendix A, since B_n doesn't enjoy the asymptotic property and so forth. Since we will eventually take as an interaction the function J of Example 1, we restrict to this case, which is much simpler because the closed form of Weber second exponential integral involves just one (modified) Bessel function I_n, instead of the correlations $J_n(r\rho)J_n(r'\rho)$.

As in (1.18), we consider the space X_k^1 of functions $v \in C^1(\mathbf{R}^+) \cap X_k^0$, such that $\sup_{r\in[1,+\infty[} |r^{k+1}v'(r)| < \infty$. The weight r^{k+1} is chosen in such a way that we can take the derivative of the asymptotics of v, and is consistent with the complete asymptotics (1.13). It is easy to see that X_k^1 is a separable Banach space, with norm

$$\|v; X_k^1\| = \|v; X_k^0\| + \sup_{r\in[0,1]} |v'(r)| + \sup_{r\in[1,+\infty[} |r^{k+1}v(r)| \tag{1.34}$$

For $k = 2$, we have $X_k^1 \subset H^1$ (the usual Sobolev space.) Let also as before $\widetilde{X}_k^1$ be the closed convex subset of X_k^1 consisting of functions $v \in C_0(\mathbf{R}^+)$, such that $v(r) + m_\beta \geq 0$, and define $\widetilde{X}_k^1([0,T]) = C^0([0,T], \widetilde{X}_k^1)$. We have :

Theorem 1.11: With the assumptions above, for all $T > 0$ small enough, and all $v_0 \in \widetilde{X}_k^1$, $k = 1, 2$, there is a unique $v \in \widetilde{X}_k^1([0, T])$ such that $u = v + m_\beta$ verifies (1.11) with $u|_{t=0} = v_0 + m_\beta$. Moreover, from any sequence $t_n \to +\infty$, we can extract a subsequence t_{n_j} such that $v(t_{n_k}, r) \to v^* \in C^1(\mathbf{R}^+)$ as $j \to \infty$ for the convergence on compact sets.

Sketch of the proof: We argue as in Theorem 1.5, showing that for small $t > 0$, $\Phi(t, r, \cdot)$ is a contraction on $\widetilde{X}_{k,v_0}^1$. This follows from the fact that A_n (in the particular case where J is a Gaussian) is a bounded operator on $\widetilde{X}_k^1$, as can be shown by using the asymptotics at infinity of I_n as in Proposition a.2 : for large rr' we replace using (1.13), $\exp[-(r^2+r'^2)/4\pi]I_n(rr'/2\pi)$ by $(rr')^{-1/2}e^{-(r-r')^2/4\pi}$, and rely on standard gaussian integral arguments.

The last part of the Theorem follows as in Corollary 1.6 from the equicontinuity of the second derivatives of $A_n u(t, r)$, as we can check by iterating (1.22), and Ascoli-Arzelà theorem. ♣.

Note that, since we proceed by extraction of subsequences, the X_k^1-ghost we obtain from the sequence $v(t_{n_j}, r)$ as a limiting orbit in the C^1-topology may not coincide with the corresponding X_k^0-ghost in the C^0-topology obtained in Corollary 1.6, even with the same initial datum. But the X_k^1-ghosts also enjoy the monotony property as in Proposition 1.10.

b) The partial dynamics, and the Barrier Lemma.

In order to cope with divergent integrals we need also study the dynamics in some finite boxes $\Lambda = \{|x| \leq \lambda\}$, for which $\lambda \to \infty$. Outside Λ, the magnetization m is frozen to a configuration m_{Λ^c} which acts as a boundary condition for the evolution inside Λ. We follow again closely the main steps of [Pr]. Define the free energy with boundary condition m_{Λ^c} as

$$\mathcal{F}(m_\Lambda | m_{\Lambda^c}) = \mathcal{F}_\Lambda(m_\Lambda) + \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dx' J(x - x')|m_\Lambda(x) - m_{\Lambda^c}(x')|^2 \tag{1.40}$$

where

$$\mathcal{F}_\Lambda(m_\Lambda) = \frac{1}{4}\int_\Lambda dx \int_\Lambda dx' J(x - x')|m_\Lambda(x) - m_\Lambda(x')|^2 + \int_\Lambda dx f_\beta(m_\Lambda(x)) \tag{1.41}$$

[Contrary to Appendix C, we have removed from the second integral the term $f(\beta m_\beta)$, which amounts to shift $\mathcal{F}_\Lambda(m_\Lambda)$ from a constant term, so long Λ is kept fixed.] Since $\int J = 1$, $\mathcal{F}(m_\Lambda | m_{\Lambda^c}) < \infty$ for all λ. We take variations of $\mathcal{F}$ inside the sector $\langle e_n \rangle$, i.e. among all radially symmetric functions of the form $m(x) = u(r)e^{in\theta}$. To simplify notations, remove subscript n from A_n, and define operators A_λ and A_λ^c by their kernel $A_\lambda(r, r') = \chi(r' \leq \lambda)A(r, r')$ and $A_\lambda^c(r, r') = \chi(r' \geq \lambda)A(r, r')$ respectively (here χ denotes the (sharp) charateristic function.) We denote also by $\widetilde{u} = u_\lambda \oplus u_\lambda^c$ the function equal to u_λ on $[0, \lambda]$ and u_λ^c outside, and write $A\widetilde{u} = (A_\lambda \oplus A_\lambda^c)(u_\lambda \oplus u_\lambda^c) = A_\lambda u_\lambda + A_\lambda^c u_\lambda^c$. We shall use twidled

u and v to refer to partial dynamics. It follows from (1.40) that Euler-Lagrange equation for $\mathcal{F}(\cdot|m_{\Lambda^c})$ restricted to the sector $\langle e_n\rangle$ is given by

$$F_1^\Lambda(\widetilde{u}) = -A\widetilde{u} + \frac{1}{\beta}\widehat{I}'(u_\lambda) = 0 \tag{1.42}$$

which is equivalent to

$$F_0^\Lambda(\widetilde{u}) = -u_\lambda + f(\beta A\widetilde{u}) = 0 \tag{1.43}$$

So we define the partial dynamics by $T_t^\Lambda u_0 = u_\lambda(t,\cdot)\oplus u_\lambda^c$ where $u_\lambda(t,\cdot)$ solves

$$\frac{du_\lambda}{dt} = -u_\lambda + f(\beta A_\lambda u_\lambda + \beta A_\lambda^c u_\lambda^c), \quad u_\lambda(0,r) = u_0(r), \quad 0\leq r\leq \lambda \tag{1.44}$$

with the same initial condition $u_0(r)$ as for the full dynamics (1.11). The integrated form of (1.44) is again

$$u_\lambda(t,r) = e^{-t}u_0(r) + \int_0^t ds e^{s-t} f(\beta A_\lambda u_\lambda(s,r) + \beta A_\lambda^c u_\lambda^c(s,r)), \quad 0\leq r\leq\lambda \tag{1.45}$$

Existence and uniqueness for (1.45), expressed in terms of $\widetilde{v} = v_\lambda \oplus v_\lambda^c$ for the decomposition $\widetilde{u} = (m_\beta + v_\lambda)\oplus(m_\beta + v_\lambda^c)$ follows as before, and $v_\lambda \in C^0\big(\mathbf{R}^+;\widetilde{W}^1(\Lambda)\cap C^1(\Lambda)\big)$ [so long as we are concerned in the first r-derivative.] Compactness of the orbits results also from the uniform boundedness of $v_\lambda(t,r)$ and $\partial_r v_\lambda(t,r)$. Namely, given any sequence $t_n\to\infty$, there is $\widetilde{u}_\lambda^*\in C^1(\Lambda)$ such that $\lim_{k\to\infty} T_{t_{n_k}}^\Lambda \widetilde{u} = \widetilde{u}^*$ for a subsequence $t_{n_k}\to\infty$.

Note that Propositions 1.9 and 1.10 also apply to partial dynamics (1.44). Decay properties of v_λ as $\Lambda\to\infty$ will follow from the "Barrier Lemma" (in the terminology of [Pr],) which is an essential tool in our analysis ; this compares the full dynamics T_t as in (1.11) with the partial dynamics T_t^Λ.

Theorem 1.12: Let $u(t,r)$ and $\widetilde{u}(t,r)$ solve respectively the full and partial dynamics (1.11) and (1.44), with same initial condition $u_0\in\widetilde{X}_2^0$. Then for all $T>0$, $u(t,r)-\widetilde{u}(t,r)\to 0$ and $\partial_r u(t,r)-\partial_r\widetilde{u}(t,r)\to 0$ uniformly for $0\leq r\leq\lambda$ and $0\leq t\leq T$, as $\lambda\to\infty$. [the r-derivative of $\widetilde{u}(t,r)$ being understood almost everywhere, namely outside $r=\lambda$.]

Proof: We proceed somewhat as in the proof of [El-BoRo,Proposition 3.5]. Recall that the restriction of T_t^Λ to $[0,\lambda]$ is given by

$$u_\lambda(t,r) = e^{-t}u_0(r) + \int_0^t dt_1 e^{-(t-t_1)} f(\beta A_\lambda u_\lambda + \beta A_\lambda^c u_\lambda^c)(t_1,r), \quad r<\lambda \tag{1.50}$$

where u_λ^c is independent of t and takes for instance the initial value u_0. [Since we are not interested in uniqueness properties of the limiting orbits, we could assume $u_\lambda^c = m_\beta$.] Denote by $u(t,r)$ the solution of (1.11) on the full space and by $\widetilde{u}(t,\cdot) = T_t^\Lambda u_0 = u_\lambda(t,\cdot) \oplus u_\lambda^c$. So $\widetilde{u}$ is piecewise continuous. We have

$$(u-\widetilde{u})(t,r) = \chi(r<\lambda) \int_0^t dt_1 e^{-(t-t_1)} g_\lambda(u,\widetilde{u})(t_1,r) + \chi(r>\lambda)(u-u_\lambda^c)(t,r) \tag{1.51}$$

with $g_\lambda(u,\widetilde{u}) = f(\beta A u) - f(\beta A_\lambda u_\lambda + \beta A_\lambda^c u_\lambda^c)$. Since $0 < f' \le 1/2$, we get

$$|g_\lambda(u,\widetilde{u})(t,r)| \le \frac{\beta}{2}|A(u-u_\lambda^c)(t,r)| = \frac{\beta}{2}|A_\lambda(u-u_\lambda)(t,r) + A_\lambda^c(u-u_\lambda^c)(t,r)| \tag{1.52}$$

all $r<\lambda$. Proposition 1.3 shows that if $u_0 - m_\beta \in \widetilde{X}_k^0$, then $u - m_\beta \in \widetilde{X}_k^0([0,T])$ for all $T>0$, and the RHS of (1.52) is well defined. Applying $A = A_\lambda \oplus A_\lambda^c$ to (1.51) we get in turn

$$A(u-\widetilde{u})(t_1,r) = A_\lambda \int_0^{t_1} dt_2 e^{t_2-t_1} g_\lambda(u,\widetilde{u})(t_2,r) + A_\lambda^c(u-u_\lambda^c)(t_1,r)$$

Using (1.52) and the fact that A_λ is positivity preserving, gives the estimate

$$|A(u-\widetilde{u})|(t_1,r) \le \frac{\beta}{2}\int_0^{t_1} dt_2 e^{t_2-t_1} A_\lambda |A(u-\widetilde{u})|(t_2,r) + |A_\lambda^c(u-u_\lambda)|(t_1,r)$$

Inserting into (1.51) we get

$$\begin{aligned}|u-\widetilde{u}|(t,r) &\le \chi(r<\lambda)\big(\frac{\beta}{2}\big)^2 \int_0^t dt_1 e^{-(t-t_1)} \int_0^{t_1} dt_2 e^{-(t_1-t_2)} A_\lambda |A(u-\widetilde{u})|(t_2,r)\\ &+ \chi(r<\lambda)\frac{\beta}{2}\int_0^t dt_1 e^{-(t-t_1)} |A_\lambda^c(u-u_\lambda^c)|(t_1,r) + \chi(r>\lambda)|u-u_\lambda|(t,r)\end{aligned}$$

Let $T^{(k)}u(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k u(t_k)$ denote the k-fold integral of u, this formula can be carried over by induction as

$$\begin{aligned}|u-\widetilde{u}|(t,r) &\le \chi(r<\lambda) e^{-t}\Big[\sum_{j=1}^{k-1}\big(\frac{\beta}{2}\big)^j T^{(j)}\big(e^{(\cdot)} A_\lambda^{j-1} |A_\lambda^c(u-u_\lambda^c)|\big)(t,r)\\ &+ \big(\frac{\beta}{2}\big)^k T^{(k)}\big(e^{(\cdot)} A_\lambda^{k-1}|A(u-\widetilde{u})|\big)(t,r)\Big] + \chi(r>\lambda)|u-u_\lambda^c|(t,r)\end{aligned} \tag{1.54}$$

We first need an estimate on $A_\lambda^c(u-u_\lambda^c)(t,r)$ for $r<\lambda$. We proceed as in Lemma c.1, using (c.6)

$$\begin{aligned}A_\lambda^c(u-u_\lambda^c)(t,r) &= (n+1)\int_\lambda^\infty dr'(u-u_\lambda^c)(t,r') \int_0^\infty d\rho \mathcal{F}\widehat{J}(\rho) J_n(r\rho) J_{n+1}(r'\rho)\\ &+ \int_\lambda^\infty dr'(u-u_\lambda^c)(t,r')\int_0^\infty d\rho\rho \mathcal{F}\widehat{J}(\rho) J_n(r\rho)\frac{d}{d\rho}J_{n+1}(r'\rho)\end{aligned} \tag{1.55}$$

We split $\int_0^\infty d\rho\mathcal{F}\widehat{J}(\rho)J_n(r\rho)J_{n+1}(r'\rho)$ into 2 parts, integrating respectively on $[0,1/\lambda]$ and $[1/\lambda,\infty[$. For the first one, we make use of the bounds $|J_n(r\rho)|\leq C(r\rho)^n$, $J_{n+1}(r'\rho)\leq C$, for the second part, of the bounds $|J_{n+1}(r'\rho)|\leq C(r'\rho)^{-1/2}$, $J_n(r\rho)\leq C$, and of the rapid decrease of $\mathcal{F}\widehat{J}(\rho)$. Altogether, we get with a new constant $C>0$:

$$|\int_0^\infty d\rho\mathcal{F}\widehat{J}(\rho)J_n(r\rho)J_{n+1}(r'\rho)|\leq C\big(\frac{1}{\lambda}\big(\big(\frac{r}{\lambda}\big)^n\vee 1\big)+\frac{1}{\sqrt{r'}}\big) \tag{1.56}$$

Next we integrate by parts $\int_0^\infty d\rho\rho\mathcal{F}\widehat{J}(\rho)J_n(r\rho)\frac{d}{d\rho}J_{n+1}(r'\rho)$, the 2 first terms can be bounded as before, for the third term $\int_0^\infty d\rho\rho r\mathcal{F}\widehat{J}(\rho)J_n'(r\rho)J_{n+1}(r'\rho)$ we split again according to $[0,1/\lambda],[1/\lambda,\infty[$, for the first integral we use $r\rho J_n'(r\rho)|\leq C\big((r\rho)^n\vee\sqrt{r\rho}\big)$, which gives the bound $C\frac{1}{\lambda}\big(\big(\frac{r}{\lambda}\big)^n\vee\sqrt{\frac{r}{\rho}}\big)$, for the last 2 ones we use $|J_n'(r\rho)|\leq C$, which gives the bound $Cr/\sqrt{r'}$. Thus

$$|\int_0^\infty d\rho\rho\mathcal{F}\widehat{J}(\rho)J_n(r\rho)\frac{d}{d\rho}J_{n+1}(r'\rho)|\leq C\big(\frac{1}{\lambda}\big(\big(\frac{r}{\lambda}\big)^n\vee\sqrt{\frac{r}{\rho}}\big)+\frac{r}{\sqrt{r'}}\big) \tag{1.57}$$

On the other hand, since the initial datum u_0 belongs to $\widetilde{X}_2^0$, given $T>0$, by Theorem 1.5 there is $C_T>0$ (also depending continuously on T) such that for all $t<T$, $|u-u_\lambda^c|(t,r)\leq C_T(1+r)^{-2}$. Inserting this estimate into (1.55), using (1.56) and (1.57) we get by integration, for a new constant $C_T>0$

$$|A_\lambda^c(u-u_\lambda^c)(t,r)|\leq C_T\frac{1}{\sqrt{\lambda}},\quad r\leq\lambda \tag{1.58}$$

Next we notice that since $e^{in\theta}A_nu(r)=J*(e^{in\cdot}u)$, and $\int J=1$, we get $|A_\lambda u|(r)\leq\sup_{[0,\lambda]}|u|$ for all $r>0$. It follows then from (1.58) that

$$A_\lambda^{j-1}|A_\lambda^c(u-u_\lambda^c)|(t,r)\leq C_T/\sqrt{\lambda},\quad j=1,2,\cdots$$

Now we have again $|A(u-\widetilde{u})(t,r)|\leq 1$ for all $t,r>$, and $A_\lambda^{k-1}|A(u-\widetilde{u})(t,r)|\leq 1$ for all k. Performing the successive integrations we find that the series in (1.54) is uniformly convergent for t in compact sets, so we can write, for $0\leq r\leq\lambda$,

$$|u-\widetilde{u}|(t,r)\leq e^{-t}\sum_{j=1}^\infty\big(\frac{\beta}{2}\big)^jT^{(j)}\big(e^{(\cdot)}A_\lambda^{j-1}|A_\lambda^c(u-u_\lambda^c)|\big)(t,r)\leq C_Te^{t(\beta/2-1)}\frac{1}{\sqrt{\lambda}}$$

which proves the first estimate on $u(t,r)-\widetilde{u}(t,r)$.

We cannot repeat this argument for the r-derivative, since ∂_r doesn't commute with A_λ [while partial derivatives ∂_{x_j} commute with convolution,] so we proceed indirectly. Take again r-derivative of (1.25), and iterate (1.22) once more to get $|\partial_r^2A_nu(t,r)|\leq\|\nabla^2J\|_1\|u\|_\infty$. By integration in the s variable, we find that $\partial_r^2 w(t,r)$, and consequently $\partial_r^2 u(t,r)$ are uniformly bounded for t in compact sets. The same conclusion holds for the partial dynamics, showing that $|\partial_r^2\big(u(t,r)-\widetilde{u}(t,r)\big)| \leq$ Const. uniformly for t in compact sets. Now we use the following well-known interpolation inequality : for any bounded intervals $K_1 \subset\subset K_2$, there is $C>0$ such that for all $f \in C^2$ we have

$$\sup_{K_1} |f'|^2 \leq C \sup_{K_2} |f| \big(\sup_{K_2} |f| + \sup_{K_2} |f''|\big)$$

Applying this to $f(r) = u(t,r) - \widetilde{u}(t,r)$ easily yields the required uniform estimate on $\partial_r u(t,r) - \partial_r \widetilde{u}(t,r)$. The theorem is proved. ♣

2. Existence of a radially symmetric minimizer.

In this Section we study some continuity properties of the free energy in finite or infinite volume ; continuity properties of the renormalized free energy are investigated in Appendix B. Next, following [Pr], we identify the limit points $\widetilde{u}^*(r)$ in the box Λ by using the excess free energy functional $\mathcal{F}(m_\Lambda | m_{\Lambda^c})$, then we identify the limit points $u^*(r)$ on the half-line, by using the renormalized energy $\mathcal{F}_{\text{ren}}(m)$ in the full space, and eventually prove Theorem 0.1.

a) Continuity properties for the free energy.

We start with general remarks on continuity properties for the free energy, which are quite close to the case of configurations valued in $[-1,1]$ as in [Pr], [AlBe],.... The only difference is that we need to assume $|m| \neq 1$, since the entropy function $I(m)$ is unbounded as $|m| \to 1$. Consider the functional

$$\mathcal{F}(m) = \frac{1}{4}\int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dy J(x-y)|m(x)-m(y)|^2 + \int_{\mathbf{R}^2} dx f_\beta(|m(x)|) \tag{2.1}$$

defined on the set $E = L^\infty(\mathbf{R}^2; B_2(0,1))$ and valued in $\mathbf{R}^+ \cup \{+\infty\}$.

Proposition 2.1: $\mathcal{F}$ is lower semicontinuous on E with respect to convergence almost everywhere.

Proof: Let $m \in E$, and $m_n \in E$ be a sequence converging a.e. to m. Define

$$g_n(x) = \frac{1}{4}\int_{\mathbf{R}^2} dy J(x-y)|m_n(x)-m_n(y)|^2 + f_\beta(|m_n(x)|)$$

and $g(x)$ similarly, with $m_n(x)$ replaced by $m(x)$. Expanding the square and using the normalization of J in L^1 we get

$$\begin{aligned} g_n(x)-g(x) &= \frac{1}{4}(|m(x)|^2 - |m_n(x)|^2) + f_\beta(|m_n(x)|) - f_\beta(|m(x)|) \\ &- \frac{1}{2}\operatorname{Re}(m_n(x)-m(x))J*\overline{m}(x) + \frac{1}{2}\operatorname{Re} m_n(x) J*(\overline{m_n}-\overline{m})(x) + \frac{1}{4}J*\big(|m_n|^2-|m|^2\big)(x) \end{aligned}$$

The first term tends to 0 a.e., so does $f_\beta(|m_n(x)|) - f_\beta(|m(x)|)$ since f_β is continuous. Since $m \in L^\infty$ and $J \in L^1$, $J * \overline{m}(x)$ is uniformly continuous and bounded, so again $(m_n(x) - m(x))J * \overline{m}(x)$ tends to 0 a.e.. In the same way, by the dominated convergence theorem, $J * (\overline{m_n} - \overline{m})(x)$, tends to 0 locally uniformly in x, and this argument equally applies to $J * \big(|m_n|^2 - |m|^2\big)(x)$, proving that the last 2 terms in the decomposition of $g_n(x) - g(x)$ above tend to 0 a.e.. Then Fatou lemma shows that

$$\liminf_{n\to\infty} \mathcal{F}(m_n) \geq \mathcal{F}(m) \tag{2.2}$$

which proves the Proposition. ♣

Thus proving the existence of a minimizer for $\mathcal{F}$ amounts to extract from every minimizing sequence a subsequence converging a.e.. The next result concerns weak lower semi-continuity of the free energy in finite volume Λ as is defined in (1.40) and (1.41). To simplify the notations, we set $m = m_\Lambda$, and $m^c = m_{\Lambda^c}$. Here $m^c = m_{\Lambda^c} \in E$ is fixed.

Proposition 2.2: If $m_n \in E$ converges weakly to m in $L^p(\Lambda)$, $1 \leq p < \infty$, then

$$\liminf_{n\to\infty} \mathcal{F}(m_n|m^c) \geq \mathcal{F}(m|m^c) \tag{2.3}$$

while, if $|m_n(x)| \leq \mu < 1$ and $m_n \to m$ a.e. in Λ, then $\mathcal{F}(m_n|m^c) \to \mathcal{F}(m|m^c)$.

Proof: Following [Pr] we write $\mathcal{F}(m|m^c) = F(m|m^c) + R(m^c)$ where

$$\begin{aligned} F(m|m^c) &= \frac{1}{\beta}\int_\Lambda dx I(m(x)) - \frac{1}{2}\operatorname{Re}\int_\Lambda dx \int_\Lambda dy J(x-y)m(x)\overline{m(y)} \\ &\quad - \operatorname{Re}\int_\Lambda dx \int_{\Lambda^c} dy J(x-y)m(x)\overline{m^c(y)} \\ R(m^c) &= \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dy J(x-y)|m^c(y)|^2 \end{aligned} \tag{2.4}$$

So $R(m^c)$ is just a constant. Because $m_n \in E$ converges weakly to m in $L^p(\Lambda)$, $\int_\Lambda dy J(x-y)\overline{m_n(y)}$ tends to $\int_\Lambda dy J(x-y)\overline{m(y)}$ as $n \to \infty$ for any $x \in \Lambda$. Moreover $\mathcal{I}_1(m) = \frac{1}{2}\operatorname{Re}\int_\Lambda dx \int_\Lambda dy\ J(x-y)m(x)\overline{m(y)}$ is weakly continuous on $L^p(\Lambda)$; namely, if m_n converges weakly to m, then $m_n(x)\overline{m_n(y)}$ converge to $m(x)\overline{m(y)}$ weakly* in $L^\infty(\Lambda\times\Lambda)$, and since $J(x-y)$ belongs to $L^1(\Lambda\times\Lambda)$, then $\mathcal{I}_1(m_n) \to \mathcal{I}_1(m)$. We get the same conclusion for the term $\mathcal{I}_2(m) = \int_\Lambda dx \int_{\Lambda^c} dy J(x-y)m_n(x)\overline{m^c(y)}$, so that (2.3) follows from the convexity of the entropy function $I(m) = \widehat{I}(|m|)$.

If $m_n \to m$ a.e. in Λ, then by the dominated convergence theorem, the last 2 terms in $F(m_n|m^c)$ tend to the corresponding ones with m instead of m_n. The same conclusion holds for the first term provided $|m_n(x)| \leq \mu < 1$ in Λ. ♣

Following [AlBe,Remark 4.8] we can use the first part of Proposition 2.2 to prove existence of a minimizer for $\mathcal{F}(\cdot|m^c)$ subject to some weakly closed constraint, since $\mathcal{F}(\cdot|m^c)$ itself is not coercive. We can also extend Proposition 2.2 to infinite volume for $\mathcal{F}$ as in (2.1), to show that E is weak* compact in $L^\infty(\mathbf{R})^2$, and $\mathcal{F}$ is weak* lower semi-continuous on E. Thus we can prove again existence of a minimizer for $\mathcal{F}$ subject to some weakly closed constraint ; but it turns out that a topological constraint such as the degree of m at infinity is not weakly closed, and thus we shall proceed another way.

b) Free energy dissipation for the partial dynamics.

This paragraph is a first step towards Theorem 0.1. The key property of the solution of (1.11) or (1.44) is that its energy decreases with time. To start with, we consider the case of finite volume. Given Λ we write again, emphasizing the dependence on θ in the sector $\langle e_n\rangle$, the solution of the partial dynamics T_t^Λ with initial value $e^{in\theta}u_0(r)$ as $\widetilde{m}(t,x)=T_t^\Lambda(e^{in\theta}u_0(r))$, using the notation $\widetilde{m}=m_\Lambda\oplus m_{\Lambda^c}=e^{in\theta}\widetilde{u}=e^{in\theta}(u_\lambda\oplus u_\lambda^c)$. Sometimes we omit to write $e^{in\theta}$. We define the free energy dissipation rate of $\widetilde{u}(t,r)$ as

$$\mathcal{I}^\Lambda(\widetilde{u})(t,r)=\frac{1}{\beta}\int_0^\lambda dr\ r\big(-\beta A\widetilde{u}+\widehat{I}'(u_\lambda)\big)\big(u_\lambda-f(\beta A\widetilde{u})\big) \tag{2.8}$$

It is easy to see that $\mathcal{F}(m_\Lambda|m_{\Lambda^c})$ is a Lyapunov function for Eqn. (1.44), i.e. $\mathcal{I}^\Lambda(\widetilde{u})\geq 0$ and

$$\mathcal{F}\big((T_t^\Lambda\widetilde{m})_\Lambda|m_{\Lambda^c}\big)-\mathcal{F}\big(m_\Lambda|m_{\Lambda^c}\big)=-\int_0^t ds\mathcal{I}^\Lambda(\widetilde{u})(s,r) \tag{2.9}$$

with $\mathcal{I}^\Lambda(\widetilde{u})=0$ iff $\widetilde{u}$ verifies (1.44). If the initial datum $\widetilde{u}$ is bounded below from 1, so is u_λ because of Proposition 1.10, and $\widehat{I}'(u_\lambda)<\infty$ everywhere. This shows that $\mathcal{I}^\Lambda(\widetilde{u})(t,r)<\infty$. We have the following

Theorem 2.3: Let the initial datum $\widetilde{m}$ be such that $\mathcal{F}(m_\Lambda|m_{\Lambda^c})<+\infty$. Then every limit point $\widetilde{u}^*(r)$ of $T_t^\Lambda\widetilde{u}$ satisfies Euler-Lagrange Eqn. (1.42)-(1.43) and

$$\mathcal{F}(e^{in\theta}u_\lambda^*(r)|m_{\Lambda^c})\leq\mathcal{F}(m_\Lambda|m_{\Lambda^c}) \tag{2.10}$$

Proof: We follow an argument of [Pr,Sect.4.2.5], essentially due to [FiMc-L]. Since $\widetilde{u}^*$ is a limit point of $T_t^\Lambda u_0$, there is a sequence $t_n\to\infty$ such that $\lim_{n\to\infty}\|T_{t_n}^\Lambda u_0-\widetilde{u}^*\|_\infty=0$. If $\widetilde{u}^*$ does not satisfy (1.42), then $\mathcal{I}^\Lambda(\widetilde{u}^*)>0$. We start from $\widetilde{u}^*$ as a new initial datum for the evolution T_t^Λ. Because $t\mapsto T_t^\Lambda(\widetilde{u}^*)$ is continuous as a map $\mathbf{R}^+\to C^0(\Lambda)$, and $\mathcal{I}^\Lambda$ is continuous (hence l.s.c) on $C^0(\Lambda)$, for $t>0$ sufficiently small we have : $\mathcal{I}^\Lambda(T_t^\Lambda\widetilde{u}^*)>0$ and hence the free energy dissipation in $[0,1]$ verifies

$$D^\Lambda(\widetilde{u}^*)=\int_0^1 dt\mathcal{I}^\Lambda(T_t^\Lambda\widetilde{u}^*)>0 \tag{2.11}$$

We shall show in a while that this implies

$$\lim_{t\to\infty}\int_0^t ds\mathcal{I}^\Lambda(T_s^\Lambda u_0)=+\infty \tag{2.12}$$

which contradicts hypothesis $\mathcal{F}(m_\Lambda|m_{\Lambda^c})<\infty$ by inequality $\leq$ in (2.9) :

$$\int_0^t ds\mathcal{I}^\Lambda(T_s^\Lambda u_0)\leq\mathcal{F}\big(m_\Lambda|m_{\Lambda^c}\big)-\mathcal{F}\big((T_t^\Lambda e^{in\theta}u_0)_\lambda|m_{\Lambda^c}\big)\leq\mathcal{F}\big(m_\Lambda|m_{\Lambda^c}\big)<\infty$$

So for $r\leq\lambda$ we have, $-\beta A\widetilde{u}^*+\widehat{I}'(u_\lambda^*)=0$, or equivalently $u_\lambda^*-f(\beta A\widetilde{u}^*)=0$, and the limiting orbit $\widetilde{u}^*$ verifies Euler-Lagrange equation (1.42). Then (2.10) easily follows from the uniform convergence on compact sets, as $t\to\infty$, of $(T_t^\Lambda e^{in\theta}u_0)_\lambda$ towards $e^{in\theta}u_\lambda^*$.

Now we need to show that (2.11) implies (2.12). Because of the continuous dependence of the initial data, we have again $\lim_{n\to\infty}\sup_{t\leq 1}\|T_t^\Lambda\widetilde{u}-T_t^\Lambda(T_{t_n}^\Lambda u_0)\|_\infty=0$. Since we work in the finite volume Λ, and $T_t^\Lambda u_0$ is bounded below from 1, this implies $\lim_{n\to\infty}D^\Lambda(T_{t_n}^\Lambda u_0)=D^\Lambda(\widetilde{u}^*)>0$. So there is $N\in\mathbf{N}$ such that $D^\Lambda(T_{t_n}^\Lambda u_0)\geq\delta>0$ for $n\geq N$. Without loss of generality (since $t_n\to\infty$) we can assume, after possibly extracting a subsequence, that $N=1$, $t_0=1$ and $t_j-t_{j-1}\geq 1$ for all $j\geq 1$, so $t_1\geq 2$ and by the group property we have

$$\int_1^{t_1}dt\mathcal{I}^\Lambda(T_t^\Lambda u_0)=\int_0^{t_1-1}dt\mathcal{I}^\Lambda(T_t^\Lambda T_1^\Lambda u_0)\geq D^\Lambda(T_1^\Lambda u_0)\geq\delta$$

By induction we get when $t\to\infty$:

$$\int_0^t ds\mathcal{I}^\Lambda(T_s^\Lambda u_0)\geq\Big(\int_1^{t_1}+\int_{t_1}^{t_2}+\cdots\Big)ds\mathcal{I}^\Lambda(T_s^\Lambda u_0)\geq\delta+\delta+\cdots\to\infty$$

which proves (2.12). ♣.

c) Free energy dissipation in infinite volume, and proof of Theorem 0.1.

We extend here the results of Paragraph b) to full dynamics. Let $m\in E$ be the initial condition for the full dynamics, such that $\mathcal{F}_{\rm ren}(m)<\infty$. More precisely, we choose $m(x)=e^{in\theta}u_0(r)$, $u_0=v_0+m_\beta\in\mathcal{W}$, $v_0\in\widetilde{X}_2^0$, with notations of Theorems b.6 and 1.5. For $\lambda>0$, denote by $\widetilde{m}=m_\Lambda\oplus m_{\Lambda^c}$, and $\widetilde{u}=u_\lambda+u_\lambda^c$ the corresponding radial parts. Here we shall let $\lambda\to\infty$, for fixed t. Using that $(T_t^\Lambda\widetilde{m})_{\Lambda^c}=m_{\Lambda^c}$, Theorem b.6 and Remark b.7 easily show that we can rewrite (2.9) as

$$\mathcal{F}_{\rm ren}\big(T_t^\Lambda\widetilde{m})\big)-\mathcal{F}_{\rm ren}\big(m\big)\leq-\int_0^t ds\mathcal{I}^\Lambda(\widetilde{u})(s,r) \tag{2.15}$$

By the Barrier Lemma, Theorem 1.12, we have $\|T_t^\Lambda\widetilde{m}-T_tm;L^\infty(\Lambda)\|=\sup_{r\in[0,\lambda]}|\widetilde{u}(t,r)-u(t,r)|\to 0$, and $\|\partial_rT_t^\Lambda\widetilde{m}-\partial_rT_tm;L^\infty(\Lambda)\|=\sup_{r\in[0,\lambda]}|\partial_r\widetilde{u}(t,r)-\partial_ru(t,r)|\to 0$, as $\lambda\to\infty$ uniformly for $t\in[0,T]$. By [lower-semi] continuity of $\mathcal{F}_{\rm ren}$, Theorem b.8

$$\liminf_{\lambda\to\infty}\big[\mathcal{F}_{\rm ren}(T_t^\Lambda\widetilde{m})-\mathcal{F}_{\rm ren}(m)\big]\geq\mathcal{F}_{\rm ren}(T_tm)-\mathcal{F}_{\rm ren}(m) \tag{2.16}$$

On the other hand, Fatou Lemma shows that

$$\liminf_{\lambda\to\infty}\int_0^t ds\mathcal{I}^\Lambda(\widetilde{u})\geq\int_0^t ds\mathcal{I}^\Lambda(T_su_0)$$

where $\mathcal{I}(u)$ is defined as in (2.8) by integrating over $r\in[0,\infty[$ and takes its values in $[0,+\infty]$. We rewrite this inequality as :

$$\limsup_{\lambda\to\infty}-\int_0^t ds\mathcal{I}^\Lambda(\widetilde{u})\leq-\int_0^t ds\mathcal{I}^\Lambda(T_su_0)$$

so by (2.15)

$$\mathcal{F}_{\mathrm{ren}}(T_tm)-\mathcal{F}_{\mathrm{ren}}(m)\leq-\int_0^t ds\mathcal{I}^\Lambda(T_su_0) \tag{2.17}$$

Now let $m^*(x)=e^{in\theta}u^*(r)$ be a limit point of T_tm, $u^*\in C^0(\mathbf{R}^+)$ as in Corollary 1.6, namely suppose there is a sequence $t_n\to\infty$ such that for any $\lambda>0$, $\lim_{n\to\infty}\sup_{r\leq\lambda}|T_{t_n}u_0(r)-u^*(r)|=0$. So this time we let $t\to\infty$, for fixed λ. If u^* does not satisfy (1.10), then there is $\lambda^*>0$ so that $\mathcal{I}^{\Lambda^*}(u^*)>0$.

We start from u^* as a new initial datum for the evolution T_t. Note that u^* has a priori no decay at infinity, but we can consider T_tu^* on $r\in[0,\lambda^*]$ by Proposition 1.8, with convergence on compact sets. Because $t\mapsto T_t(u^*)$ is continuous as a map $\mathbf{R}^+\to C^0(\Lambda^*)$, and $\mathcal{I}^{\Lambda^*}$ is continuous (hence l.s.c) on $C^0(\Lambda^*)$, for $t>0$ sufficiently small we have : $\mathcal{I}^{\Lambda^*}(T_tu^*)>0$ and hence the free energy dissipation for $(t,r)\in[0,1]\times[0,\lambda^*]$ verifies

$$D^*(u^*)=\int_0^1 dt\mathcal{I}^{\Lambda^*}(T_tu^*)>0 \tag{2.19}$$

As in the proof of Proposition 2.3, we shall show that this implies

$$\lim_{t\to\infty}\int_0^t ds\mathcal{I}^{\Lambda^*}(T_su_0)=+\infty \tag{2.20}$$

On the other hand, inequality (2.17) yields $\int_0^t ds\mathcal{I}^{\Lambda^*}(T_su_0)\leq\int_0^t ds\mathcal{I}(T_su_0)\leq\mathcal{F}_{\mathrm{ren}}(m)-\mathcal{F}_{\mathrm{ren}}(T_tm)$. So by the discussion at the end of Appendix B, there is $C>0$ such that $\mathcal{F}_{\mathrm{ren}}(T_tm)\geq-C$ for all $t>0$ large enough, and we see that (2.20) contradicts the hypothesis $\mathcal{F}_{\mathrm{ren}}(m)<\infty$.

Now we prove that (2.19) implies (2.20). Since $T_{t_n}u_0(r)\to u^*$ uniformly for $(t,r)\in[0,1]\times[0,\lambda^*]$ as $n\to\infty$, and the flow T_t depends continuously on the initial data, we see that $T_t(T_{t_n}u_0)(r)-T_tu^*(r)\to0$ as $n\to\infty$, uniformly for $(t,r)\in[0,1]\times[0,\lambda^*]$, and by Fatou Lemma,

$$\lim_{n\to\infty}D^*(T_{t_n}u_0)\geq\int_0^1 dt\mathcal{I}^{\Lambda^*}(T_tu^*)>0$$

So we may assume that $D^*(T_{t_n}u_0) \geq \delta > 0$ for all n large enough, and the proof goes exactly as in Proposition 2.3. This shows (2.20) and brings the proof of Theorem 0.1 to an end. ♣

3. Linear stability.

To start with, we recall from [OvSi1] some well known facts concerning symmetry breaking of Eqn. (1.8) or (1.9). The symmetry group G for these equations is given by $G = \mathbf{R}^2 \times O^+(2) \times U(1) \times \Gamma$, where $\mathbf{R}^2$ acts as translations $m(x) \to m(x-h)$, $h \in \mathbf{R}^2$, $O^+(2)$ as rotations $m(x) \to m(R^{-1}x))$, $R \in O^+(2)$, $U(1)$ as gauge transformation $m(x) \to \lambda m(x)$, $\lambda \in U(1) \approx S^1$, and Γ as "conjugation of charge" $m(x) \to \overline{m(x)}$.

By the symmetry group G_m of a solution m, we mean the largest subgroup of G which leaves m fixed. Then the part of G broken by m is the coset G/G_m. If H is a one-parameter sub-group of G, we say that H is *preserved* (resp. *broken*) by m if $h(m) = m$ for all $h \in H$ (resp. $h(m) \neq m$ for all $h \in H, h \neq \mathrm{Id}$.)

The subgroup of translations is never preserved by m, unless m is a constant. The symmetry group of a radially symmetric solution, i.e. $m(x) = e^{in\theta}u(r)$ is the discrete subgroup of $O^+_{k/n}(2) \subset O^+(2)$ of rotations by the angles $2k\pi/n, k \in \mathbf{Z}$. Thus, m breaks the translation group, the rotation subgroup $O^+(2)/O^+_{k/n}(2)$, and the charge group. The symmetry group for an equation of the form $F(m) = 0$ allows to find elements in the kernel of its linearization around some point m, i.e. solutions of $\langle dF(m), \xi\rangle = 0$. Namely, let m be a solution of $F(m) = 0$, breaking a one parameter subgroup $g(s) \in G$ (the symmetry group of this equation). Let τ be the generator of $g(s)$. Then $\xi = \tau m$ solves the linearized equation $\langle dF(m), \xi\rangle = 0$.

a) Linearisation of $\mathcal{F}_{\mathrm{ren}}$ around a radially symmetric solution.

Next we examine the Hessian of $\mathcal{F}_{\mathrm{ren}}$ and look for relations between the gradients of F_1 and F_0 defined in (1.8), (1.9). We take advantage of the fact that F_j, $j = 0, 1$, are real analytic functions, real for real m, to write $F_j(m)$ instead of $F_j(m, \overline{m})$, and we will denote also $F_j(\overline{m}) = \overline{F_j(m, \overline{m})} = F_j(\overline{m}, m)$, $\frac{\partial \overline{F_1}}{\partial m}(m) = \frac{\partial F_1}{\partial \overline{m}}(\overline{m})$, etc ... Following [OvSi], we express the Hessian of $\mathcal{F}_{\mathrm{ren}}$ in these complex variables as

$$\mathrm{Hess}\,\mathcal{F}_{\mathrm{ren}} = \begin{pmatrix} \frac{\partial^2 \mathcal{F}}{\partial m \partial \overline{m}} & \frac{\partial^2 \mathcal{F}}{\partial \overline{m}^2} \\ \frac{\partial^2 \mathcal{F}}{\partial m^2} & \frac{\partial^2 \mathcal{F}}{\partial m \partial \overline{m}} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial m} & \frac{\partial F_1}{\partial \overline{m}} \\ \frac{\partial \overline{F_1}}{\partial m} & \frac{\partial \overline{F_1}}{\partial \overline{m}} \end{pmatrix} \tag{3.1}$$

We have $\mathrm{Hess}\,\mathcal{F}_{\mathrm{ren}}(m) = \nabla_m F_1(m)$ (the gradient in the real sense) and similarly, we define $\nabla F_0(m)$. Introduce the notations $z = \beta J * m$, $\phi_0(z) = f(|z|)\frac{z}{\overline{z}}$, $\phi_1(m) = \widehat{I}'(|m|)\frac{m}{|m|}$, so that ϕ_0 and ϕ_1 are inverse from each other.

Lemma 3.1: With the notations above

$$-\beta\nabla_z\phi_0(z)\nabla_m F_1(m) = -\mathrm{Id} + \beta\nabla_z\phi_0(z)J * \cdot = \nabla_m F_0(m) \tag{3.2}$$

and in particular $\big(\beta\nabla_z\phi_0(z)\big)^{-1}=\nabla_m F_1(m)+J*$.

Proof: Again, we write $\phi_1(m)$ for $\phi_1(m,\overline{m})$, $\phi_1(\overline{m})$ for $\phi_1(\overline{m},m)$, and similarly for ϕ_0. The "upper-left" matrix element of $\nabla\phi_0(z)\nabla F_1(m)$ is given by

$$a_1=-\frac{\partial\phi_0}{\partial z}(z)J*\cdot+\frac{1}{\beta}\big[\frac{\partial\phi_0}{\partial z}(z)\frac{\partial\phi_1}{\partial m}(m)+\frac{\partial\phi_0}{\partial\overline{z}}(z)\frac{\partial\phi_1}{\partial\overline{m}}(\overline{m})\big]$$

On the other hand, differentiating the identity $\phi_0\circ\phi_1=\mathrm{Id}$ we get

$$1=\partial_m(\phi_0\circ\phi_1)(m)=\frac{\partial\phi_0}{\partial z}(z)\frac{\partial\phi_1}{\partial m}(m)+\frac{\partial\phi_0}{\partial\overline{z}}(z)\frac{\partial\phi_1}{\partial\overline{m}}(\overline{m})$$

which leads to $\beta a_1=1-\beta\frac{\partial\phi_0}{\partial z}(z)J*\cdot$. All other matrix elements can be handled of this sort, using $\partial_{\overline{m}}(\phi_0\circ\phi_1)(m)=0$, and two similar identities, obtained after permuting z with $\overline{z}$, m with $\overline{m}$. So we proved

$$-\beta\nabla\phi_0(z)\nabla F_1(m)=-\mathrm{Id}+\beta\begin{pmatrix}\frac{\partial\phi_0}{\partial z}(z)&\frac{\partial\phi_0}{\partial\overline{z}}(z)\\ \frac{\partial\phi_0}{\partial\overline{z}}(\overline{z})&\frac{\partial\phi_0}{\partial z}(\overline{z})\end{pmatrix}J*\cdot\tag{3.3}$$

and the Lemma easily follows. ♣

A direct computation also shows $\frac{\partial\phi_0}{\partial z}(z)=\frac{1}{2}\big(\frac{f(|z|)}{|z|}+f'(|z|)\big)$. Using recursion formulas between the derivatives of the modified Bessel functions I_0 and I_1, we find for $f=I_1/I_0$: $f'(t)=1-f^2(t)-\frac{f(t)}{t}$, so

$$\frac{\partial\phi_0}{\partial z}(z)=\frac{1}{2}(1-f^2(|z|))\tag{3.4}$$

Similarly

$$\frac{\partial\phi_0}{\partial\overline{z}}(z)=\frac{z}{2\overline{z}}(1-f^2(|z|)-2\frac{f(|z|)}{|z|})\tag{3.6}$$

so that setting $b(|z|)=1-f^2(|z|)-2\frac{f(|z|)}{|z|}$, we have

$$\nabla_z\phi_0(z)=\frac{1}{2}\begin{pmatrix}1-f^2(|z|)&\frac{z}{\overline{z}}b(|z|)\\ \frac{\overline{z}}{z}b(|z|)&1-f^2(|z|)\end{pmatrix}\tag{3.7}$$

It follows that $\nabla_z\phi_0(z)$ is hermitean, and

$$\det\nabla\phi_0(z)=a(|z|)=f'(|z|)\frac{f(|z|)}{|z|}\tag{3.8}$$

We notice that $1-f^2(|z|)>0$, and $b(|z|)=f'(|z|)-\frac{f(|z|)}{|z|}\leq 0$ with equality only at $z=0$, since $f(0)=0$ and f is strictly concave on $[0,+\infty[$. Now by Lemma 3.1

$$\nabla F_0(m)=-\mathrm{Id}+\frac{\beta}{2}\begin{pmatrix}1-f^2(|z|)&\frac{z}{\overline{z}}b(|z|)\\ \frac{\overline{z}}{z}b(|z|)&1-f^2(|z|)\end{pmatrix}J*\cdot\tag{3.9}$$

Let now m be a radially symmetric solution of $d\mathcal{F}_{\text{ren}} = 0$, as in Theorem 0. If $m(x) = e^{in\theta}u(r)$ solves (1.8), then $f(|z|) = u(r)$, and we have $b(|z|) = b(r) = 1 - u^2(r) - 2\frac{u(r)}{\widehat{I'} \circ u(r)}$, $a(r) = a(|z|) = f' \circ f^{-1} \circ u(r)\frac{u(r)}{f^{-1} \circ u(r)}$ (here we have used the relation $|m| = f(|z|)$ to denote, somewhat incorrectly, $a(|z|)$ by $a(r)$, $b(|z|)$ by $b(r)$.) Then Lemma 3.1 shows again

$$\text{Hess}\,\mathcal{F}_{\text{ren}}(m) = \nabla_m F_1(m) = -\frac{1}{2\beta}B(r,\theta) + J * \cdot \tag{3.10}$$

with

$$B(r,\theta) = \frac{1}{a(r)}\begin{pmatrix} 1 - u^2(r) & -e^{2in\theta}b(r) \\ -e^{-2in\theta}b(r) & 1 - u^2(r) \end{pmatrix} \tag{3.11}$$

(cf. [OvSi1, formula (7.7)] for Ginzburg-Landau equation.) As in [OvSi] we have the

Lemma 3.2: ∇F_1 is symmetric for the scalar product $\langle \xi, \eta \rangle = \text{Re} \int \overline{\eta}\xi$. In other words, $\text{Re} \int \overline{\eta} \nabla F_1(\xi) = \text{Re} \int \overline{\nabla_1(\eta)}\xi$, where $\nabla F_1(\xi)$ is a shorthand for $\langle \nabla F_1(x), (\xi, \overline{\xi})\rangle$.

Now we make a Fourier analysis of ∇F_1. Taking polar coordinates as above, write $\xi = \sum_{k\in\mathbf{Z}} \xi_k(r)e^{ik\theta}$, $\xi_k \in \mathbf{C}$, and expand $J * \cdot$ in terms of A_k. To account for the phase factors $e^{\pm 2in\theta}$ in (3.11) we make a shift of indices and introduce the mapping

$$\pi : (\xi, \overline{\xi}) = \big((\xi_k, \overline{\xi}_k)\big)_{k\in\mathbf{Z}} \mapsto \widehat{\xi} = \big((\xi_k, \overline{\xi}_{2n-k})\big)_{k\in\mathbf{Z}}$$

which is unitary if the target space is endowed with the inner product

$$\langle \widehat{\xi}, \widehat{\eta} \rangle = \text{Re}\langle \widehat{\xi}_n, \widehat{\eta}_n \rangle + \text{Re} \sum_{k>n} \langle \begin{pmatrix} \xi_k \\ \overline{\xi}_{2n-k} \end{pmatrix}, \begin{pmatrix} \eta_k \\ \overline{\eta}_{2n-k} \end{pmatrix} \rangle \tag{3.13}$$

We then define the real-linear operator $\widehat{\nabla}_m F_1$ on vector-valued functions $\widehat{\xi}$ by $\widehat{\nabla}_m F_1 = \pi \nabla_m F_1 \pi^*$. As in [OvSi] we can easily show that operator $\widehat{\nabla}_m F_1$ is block-diagonal of the form

$$\langle \widehat{\nabla}_m F_1, \widehat{\xi} \rangle = \big(\langle \widehat{\nabla}_m F_1^{(k)}, \begin{pmatrix} \xi_k \\ \overline{\xi}_{2n-k} \end{pmatrix} \rangle\big)_{k\geq n}$$

where

$$\widehat{\nabla}_m F_1^{(k)} = \begin{pmatrix} A_k - \frac{1}{2\beta a(r)}(1 - u^2(r)) & \frac{b(r)}{2\beta a(r)} \\ \frac{b(r)}{2\beta a(r)} & A_{2n-k} - \frac{1}{2\beta a(r)}(1 - u^2(r)) \end{pmatrix} = L_k \tag{3.14}$$

So we have reduced the problem of linear stability around the radially symmetric solution $m(x) = e^{in\theta}u(r)$ to the study of the spectrum of the family of operators $(L_k)_{k\geq n}$. By the discussion at the beginning of this Section, we already know some zero modes, corresponding to breaking of the rotation, gauge or translation symmetry. However, these functions are not in L^2.

b) Spectral properties of L_n.

Conjugating with $R = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, we see that L_n is unitary equivalent to

$$\widetilde{L}_n = \begin{pmatrix} A_n - \frac{\widehat{I}'(u)}{\beta u} & 0 \\ 0 & A_n - \frac{\widehat{I}''(u)}{\beta} \end{pmatrix} \tag{3.15}$$

Let $V(r) = \frac{\widehat{I}'(u)}{\beta u} > 0$, $W(r) = \frac{\widehat{I}''(u)}{\beta} > 0$. These are smooth functions on $\mathbf{R}^+$. Since $A_n u - \frac{1}{\beta}\widehat{I}'(u) = 0$, $U = \binom{u}{0}$ is a solution of $\widetilde{L}_n U = 0$ (the zero mode is due to breaking the gauge group) but of course $u \notin L^2$. First we look at the spectrum of A_n.

Lemma 3.3: If $\mathcal{F}\widehat{J} \geq 0$ and $\widehat{J} \geq 0$ (i.e. J is non negative definite in the sense of [FrTo],) then $]0,1[\subset \sigma_c(A_n) \subset \sigma(A_n) \subset [0,1]$. Moreover $\mathrm{Ker}\, A_n = 0$.

Proof: Recall $A_n = H_n \mathcal{F}\widehat{J} H_n$, so $(A_n - \lambda)^{-1} = -\frac{1}{\lambda} H_n \big(1 - \frac{\mathcal{F}\widehat{J}}{\lambda}\big)^{-1} H_n$. So $(A_n - \lambda)^{-1}$ is bounded in operator norm for $\lambda < 0$, if $\mathcal{F}\widehat{J} \geq 0$. On the other hand, $\mathcal{F}\widehat{J}(\rho) \leq 1$ since $\widehat{J} \geq 0$ and $\int J = 1$, so when $\lambda > 1$, $\big(1 - \frac{\mathcal{F}\widehat{J}(\rho)}{\lambda}\big)^{-1}$ is a Neuman series converging in operator norm. It follows that $\sigma(A_n) \subset [0,1]$. Conversely, let $\lambda \in]0,1[$. Since $\widehat{J}(0) = 1$ and $\mathcal{F}\widehat{J}(\rho) \to 0$ as $\rho \to \infty$, there is $\rho_\lambda > 0$ such that $\mathcal{F}\widehat{J}(\rho_\lambda) = \lambda$, so take $v \in L^2(\mathbf{R}^+; rdr)$ such that $H_n v(\rho) = \chi(\rho_\lambda - \varepsilon < \rho < \rho_\lambda + \varepsilon)$, where $\varepsilon > 0$ is small enough. It is clear that $(A_n - \lambda)^{-1} v \notin L^2$, which shows that $A_n - \lambda$ is not surjective.

At last, let $\psi \in L^2(\mathbf{R}^+; rdr)$ such that $A_n \psi = 0$. Then $\mathcal{F}\widehat{J}(\rho) H_n \psi(\rho) = 0$ a.e. and this implies in turn that $H_n \psi(\rho) = 0$ a.e. for we cannot have $\mathcal{F}\widehat{J}(\rho) = 0$ on a set of positive measure since $\widehat{J}$ is smooth and $\widehat{J} \geq 0$. So $\psi = 0$ and $\mathrm{Ker}\, A_n = 0$. ♣

Consider now $A_n - V(r)$, and $A_n - W(r)$. We discuss first some properties of V and W. Convexity of the entropy function $\widehat{I}$ on $[0,1]$ and $\widehat{I}'(0) = 0$ imply that $\widehat{I}''(u) \geq 0$, and $V'(u) = \widehat{I}''(u) - \frac{\widehat{I}'(u)}{u} \geq 0$. In the same way, it is easy to see that $\widehat{I}'''(u) \geq 0$. Recall also that $\widehat{I}''(0) = 2$, and m_β satisfies the equation $\widehat{I}'(m_\beta) = \beta m_\beta$. Although this will not be needed, we mention for completeness that $\widehat{I}''(m_\beta) = \big(\frac{I_0''}{I_0}(\beta m_\beta) - m_\beta^2\big)^{-1}$.

On the other hand we learned from Sect.2 that u increases from 0 to m_β on $\mathbf{R}^+$. Altogether, this shows that V increases from $2/\beta$ to 1 on $r \in [0, +\infty[$ for $\beta > 2$ (recall that there is nothing to prove when $\beta \leq 2$,) and W increases from $2/\beta$ to $\widehat{I}''(m_\beta)$.

So in the sense of self-adjoint operators, we have $A_n - 1 \leq A_n - V(r) \leq A_n - 2/\beta$, and Lemma 3.3 easily shows that $\sigma(A_n) \subset [-1, 1 - 2/\beta]$. For convenience we shift the whole spectrum by 1 by changing V into $\widetilde{V} = V - 1$, and consider now $A_n - \widetilde{V}$. For a suitable interaction J we shall prove that the spectrum of $A_n - V$ is purely continuous by using a positive commutator. This is actually a simple variant of Mourre estimates. More precisely, we assume that $\widehat{J}(r) = (2\pi)^2 e^{-r^2/4\pi}$, or equivalently $\mathcal{F}\widehat{J}(\rho) = e^{-\pi\rho^2}$ (see Example 1 of Sect.1.)

Lemma 3.4: With J as above we have $[J * \cdot, x\partial_x + \partial_x x] = -4\pi\Delta J$, or in the $\langle e_n\rangle$-sector, $[A_n, r\partial_r + \partial_r r] = 4\pi H_n(\rho^2 e^{-\pi\rho^2})H_n$.

Proof: We compute for a test function φ, $[J*\cdot, x\partial_x+\partial_x x]\varphi(x) = -2\int dx J(x-y)\langle x-y, \nabla\varphi(y)\rangle$. When $J(x) = (2\pi)^2 e^{-x^2/4\pi}$, $-2J(x-y)(x-y) = 4\pi J'(x-y)$, so Green formula shows that $[J*\cdot, x\partial_x + \partial_x x] = -4\pi\Delta J$. Taking Fourier transform and restricting to the $\langle e_n\rangle$-sector gives the Lemma. ♣

Denote by $D = r\partial_r + \partial_r r$ the generator of dilations, and $C_n = 4\pi H_n(\rho^2 e^{-\pi\rho^2})H_n$. So C_n is a bounded, positive operator in the L^2 sense (and also positivity improving as we can show by differentiating (1.12) with respect to p at $p = \pi$; also C_n cuts off both low and high frequencies. These properties however, will not be used in the sequel .)

On the other hand, $[-\widetilde{V}, D] = 2r\widetilde{V}'(r)$, which is also a positive (and positivity improving) operator, so we get :

$$[A_n - \widetilde{V}, D] = C_n + 2r\widetilde{V}'(r) \tag{3.17}$$

From this and the energy identity

$$\big([A_n - \widetilde{V}, D]\psi|\psi\big) = (C_n\psi|\psi) + (2r\widetilde{V}'(r)\psi|\psi) \tag{3.18}$$

it follows that $\psi \in \mathcal{D}(D) = \{\psi \in L^2(\mathbf{R}^+; rdr) : r\psi'(r) \in L^2\}$ cannot be an eigenfunction of $A_n - \widetilde{V}$, for otherwise it would be supported at $r = 0$ (and the range of its frequencies would reduce to $\rho = 0$.) But since D is not bounded on L^2 we shall first perform a regularization. This is simpler than usual (with the laplacian instead of a convolution operator) because A_n itself is bounded. Namely as in [Mo] we use the following :

Lemma 3.5: For real $\mu \neq 0$, let $R_\mu = (1 + D/\mu)^{-1}$. Then for $|\mu|$ large enough, R_μ is uniformly bounded on $L^2(\mathbf{R}^+; rdr)$, and $\text{s-lim}_{\mu\to\infty} R_\mu = \text{Id}$.

Remark: Actually, as in [Mo] we could strengthen the conclusion of the Lemma by replacing $L^2(\mathbf{R}^+; rdr)$ by $\mathcal{H}_k$, $k = 0, \pm1, \pm2$. Here $\mathcal{H}_0 = L^2$, $\mathcal{H}_2$ denotes the domain of a second order differential operator (laplacian) on the half-line $\Delta_\alpha = r^{-2}\big((r\partial_r)^2 + r\partial_r - \alpha\big)$, $\alpha > 0$, i.e. $\mathcal{H}_2 = \{\psi \in L^2 : \Delta_\alpha\psi \in L^2\}$, $\mathcal{H}_1$ is the closure of $C_0^\infty(\mathbf{R}^+)$ for the Sobolev norm $\|\psi\|_{1,\alpha} = \big(\int_0^\infty drr(|\psi|^2 + |\partial_r\psi|^2 + \frac{\alpha}{r^2}|\psi|^2)\big)^{1/2}$, and are $\mathcal{H}_{-1}, \mathcal{H}_{-2}$ the dual spaces. Actually some care is needed because of the singularity at $r = 0$. We shall not use this however, but it could be also useful, if we want to drop the assumption about the existence of an asymptotic of $u(r)$ as $r \to \infty$, to work also on weighted spaces of the form $\langle r\rangle^k L^2(\mathbf{R}^+; rdr)$.

Given Lemma 3.5, we can prove absence of eigenvalues for $A_n - \widetilde{V}$ as follows. Substituting $D_\mu = DR_\mu$ for D in the LHS of (3.18) we get : $\big((A_n - \widetilde{V})D_\mu\psi|\psi\big) - \big(D_\mu(A_n - \widetilde{V})\psi|\psi\big) = \big(D_\mu\psi|(A_n - \widetilde{V})\psi\big) = 0$ since $(A_n - \widetilde{V})$ is self-adjoint and $D_\mu\psi \in \mathcal{D}(A_n - \widetilde{V}) = L^2$. Now by the definition of R_μ, we have $[A_n - \widetilde{V}, D_\mu] = R_\mu[A_n - \widetilde{V}, D]R_\mu$. Using that s-$\lim_{\mu\to\infty} R_\mu = \mathrm{Id}$ and $[A_n, D]$ is bounded gives s-$\lim_{\mu\to\infty} R_\mu[A_n, D]R_\mu = [A_n, D]$. We are left with $R_\mu[A_n, \widetilde{V}]R_\mu$. Taking r-derivative of (1.10) gives $u'(r) = \beta f'(\beta A_n u(r))(A_n u)'(r)$, and by the explicit representation (1.12) of A_n

$$ru'(r) = (2\pi)^{-2}\beta f'(\beta A_n u(r))\big(r[A_n, r]u(r) - 2\pi n A_n u(r)\big) \tag{3.19}$$

The second term on the RHS is uniformly bounded for $r \in \mathbf{R}^+$, since A_n is bounded on L^∞. We know by Theorem 1.11 that $u \in C^1$. According to hypothesis of Theorem 0.2, we shall also assume that u has an asymptotics of the form $u(r) = m_\beta + \frac{c}{r} + \cdots$ as $r \to \infty$. Since A_n has the asymptotic property with vanishing subprincipal symbol, $r[A_n, r]$ is bounded on $m_\beta + X_2^0$, so $r \mapsto r[A_n, r]u(r)$ is bounded on $\mathbf{R}^+$. It follows from (3.19) that the multiplication by $r\widetilde{V}'(r)$ is bounded on L^2, and again by Lemma 3.5, s-$\lim_{\mu\to\infty} R_\mu[A_n, \widetilde{V}]R_\mu = [A_n, \widetilde{V}]$. So we get $\big([A_n - \widetilde{V}, D_\mu]\psi|\psi\big) \to \big([A_n - \widetilde{V}, D]\psi|\psi\big)$ as $\mu \to \infty$, and looking at the RHS of (3.18) gives that $\psi = 0$.

Absence of singular spectrum follows also from this regularization and Putnam-Kato theorem (see [ReSi,Sect.XIII.7].)

By the discussion before Lemma 3.4, the same argument applies to $A_n - W(r)$, so by (3.15), L_n has purely continuous spectrum, which eventually achieves proving Theorem 0.2.

c) Remarks on the linear stability of higher modes.

Spectral analysis of operators L_k, $k \geq n+1$ is much harder, and we shall content to write the formula for L_{n+1}. For the scalar 1-d Kac model, it is known (see [Pr,Sect.(6.3.1)] that if L denotes the corresponding linear operator around an instanton $m(x)$, i.e. a solution to Euler-Lagrange equation standing for (1.8) or (1.9), then L is bounded, its spectrum lies in $\mathbf{R}^-$, and L has a 1-d kernel generated by $m'(x)$. Moreover there is a spectral gap $\omega > 0$, namely $(\psi|L\psi) \leq -\omega\|\psi\|^2$ for all ψ such that $(m'|\psi) = 0$. This relies on the fundamental energy identity

$$(\psi|L\psi) = -\frac{1}{2}\int dxdy J(x-y)m'(x)m'(y)\Big(\frac{\psi(x)}{m'(x)} - \frac{\psi(y)}{m'(y)}\Big)^2$$

which of course doesn't hold in our case. Instead we write, after conjugating with the reflection matrix R as before, operator L_{n+1} in the form :

$$\widetilde{L}_{n+1} = \begin{pmatrix} A_{n+1} + A_{n-1} - V(r) & A_{n-1} - A_{n+1} \\ A_{n-1} - A_{n+1} & A_{n+1} + A_{n-1} - W(r) \end{pmatrix}$$

The vector $U(r) = \binom{nu(r)/r}{u'(r)}$ corresponds to the translation mode, and solves $\widetilde{L}_{n+1}U(r) = 0$. With the Gaussian J, naive considerations on the matrix $[\widetilde{L}_{n+1}, D]$ suggest it could enjoy positivity properties, but still allowing for a non trivial kernel.

Appendix

A) Continuity properties of A_n.

We analyze in this section the action of A_n on X_k^0 and Y_k^0.

Lemma a.1: Assume $J \in \mathcal{S}(\mathbf{R}^2)$ (the Schwartz space of rapidly decreasing functions together with all derivatives). Let χ be a smooth cut-off equal to 0 on $[0, 1/2]$ and to 1 on $[1, \infty[$, then if u is continuous on the half-line, we have

$$(a.3) \qquad \sup_{r\leq 1} |A_n(1-\chi)u(r)| \leq C \sup_{r'\leq 1} |u(r')|$$

$$(a.4) \qquad \sup_{r\geq 1} r^k |A_n(1-\chi)u(r)| \leq C \sup_{r'\leq 1} |u(r')|, \quad k = 0, 1, 2.$$

$$(a.5) \qquad \sup_{r\leq 1} |A_n \chi u(r)| \leq C \sup_{r'\geq 1/2} |u(r')|$$

Moreover, (a.4) holds true for $k = 5/2$ and for $k = 3$ when $n \geq 2$.

Proof: For (a.3), we simply use the fact that Bessel functions (together with all their derivatives,) are bounded on $\mathbf{R}^+$ (for example $J_0(x)| \leq 1$, and $|J_n(x)| \leq 1/\sqrt{2}$, all $n \geq 1$), and $\mathcal{F}\widehat{J}(\rho)$ rapidly decreasing. Next we check (a.4). For $k = 0$, the same argument apply. When $k = 1$, from the resursion formula for Bessel functions,

$$(a.6) \qquad (n+1)J_{n+1}(x) = x\big(J_n(x) - J'_{n+1}(x)\big)$$

we get

$$(a.7) \qquad \begin{aligned} rA_n(1-\chi)u(r) = (n+1)\int_0^\infty dr' r'(1-\chi)u(r') \int_0^\infty d\rho J_{n+1}(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho) + \\ + \int_0^\infty dr' r'(1-\chi)u(r') \int_0^\infty d\rho r\rho J'_{n+1}(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho) \end{aligned}$$

By the uniform bound on Bessel functions we just recalled, the first term is estimated by the RHS of (a.4). For the second term, we integrate by parts with respect to ρ in the second integral, using that $J \in \mathcal{S}(\mathbf{R}^2)$, which yields the same conclusion. So (a.4) is proved when $k = 1$. Consider at last the case $k = 2$. Denote by $b_1(r, r')$ and $b_2(r, r')$ the two ρ-integrals in (a.7). We compute

$$(a.8) \qquad \begin{aligned} rb_1(r, r') = (n+1)(n+2)\int_0^\infty \frac{d\rho}{\rho} J_{n+2}(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho) + \\ + (n+1)\int_0^\infty d\rho J_n(r'\rho)\mathcal{F}\widehat{J}(\rho)\frac{d}{d\rho}J_{n+2}(r\rho) \end{aligned}$$

To estimate the first term, we split the integration over $[0,1]$ and $[1,\infty]$, on $[0,1]$ we use that $J_n(r'\rho) \leq \text{Const.}(r'\rho)^n$ since r' is bounded on supp $1-\chi$. Performing the integration, this brings a term $\mathcal{O}(r'^n)$. The integral on $[1,\infty[$ can be directly evaluated. We estimate the second term in $rb_1(r,r')$, and also $rb_2(r,r')$ in the same way. So we proved (a.4) for $k=0,1,2$ and for $k=3$ the same arguments apply when $n \geq 2$.

Let us consider the case $k=5/2$. Call $b_{11}(r,r')$ and $b_{12}(r,r')$ the 2 terms in (a.8), and compute

$$(a.8) \qquad r^{1/2}b_{11}(r,r') = (n+1)(n+2)\int_0^\infty \frac{d\rho}{\rho^{3/2}}(r\rho)^{1/2}J_{n+2}(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho)$$

Since, for all n

$$J_n(x) \sim \sqrt{2}(\pi x)^{-1/2}\cos(x-\pi n/2-\pi/4),\ x\to\infty$$

$(r\rho)^{1/2}J_{n+2}(r\rho) = \mathcal{O}(1)$ uniformly for $r\rho>0$. We split the integral in (a.8) as $\int_0^1$ and $\int_1^\infty$. Using $J_n(r'\rho) \leq \text{Const.}(r'\rho)^n$ for $r' \in \operatorname{supp}\chi$, $\rho \in [0,1]$, the $\int_0^1$-integral contributes for a constant times $r'^n \int_0^1 d\rho\rho^{n-3/2} = \mathcal{O}(r'^n)$, while the $\int_0^\infty$-integral contributes for a constant. Hence $r^{1/2}\int_0^\infty dr'r'(1-\chi)u(r')b_{11}(r,r') \leq C\sup_{r'\in[0,1]}|u(r')|$. The same arguments show this is the case of the corresponding term with $r^{1/2}b_{12}(r,r')$, and also of those arising from $rb_2(r,r')$. So (a.4) holds true when $k=5/2$.
We proceed analogously for (a.5), and compute $A_n((\cdot)^3v)(r)$, where $v(r') = \chi(r')r'^{-3}u(r')$, so that $|v(r')| \leq \chi(r')r'^{-3}\sup_{r'\geq 1/2}|u(r')|$, and $\chi(r')r'^{-3}$ is integrable.

We estimate first $a_1(r,r') = r'\int_0^\infty d\rho\rho J_n(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho)$ using (a.6), as $r'\rho J_n(r'\rho) = (n+1)J_{n+1}(r'\rho) + \rho\frac{d}{d\rho}J_{n+1}(r'\rho)$ which gives accordingly $a_1(r,r') = a_{11}(r,r') + a_{12}(r,r')$. In $a_{12}(r,r')$ we integrate by parts as before, which gives bounded terms for $r \leq 1$. Multiply again by r', and write $a_2(r,r') = r'a_1(r,r') = a_{21}(r,r') + a_{22}(r,r')$, with $a_{21}(r,r') = (n+1)(n+2)\int_0^\infty \frac{d\rho}{\rho}J_n(r\rho)J_{n+2}(r'\rho)\mathcal{F}\widehat{J}(\rho)$, and $a_{22}(r,r')$, a less singular term, containing also derivatives of $J_{n+2}(r'\rho)$, which we integrate by parts. We multiply a last time by r', use again (a.6) and write $a_3(r,r') = r'a_2(r,r') = a_{31}(r,r') + a_{32}(r,r')$, with $a_{31}(r,r') = (n+1)(n+2)(n+3)\int_0^\infty \frac{d\rho}{\rho^2}J_n(r\rho)J_{n+3}(r'\rho)\mathcal{F}\widehat{J}(\rho)$, and a_{32} a less singular term, containing also derivatives of $J_{n+3}(r'\rho)$. We examine a_{31} by splitting the integration over $[0,\infty[$ into $[0,r'^{-1}]$, $[r'^{-1},1]$, and $[1,\infty]$. On $[0,r'^{-1}]$, we use the estimates $J_n(r\rho) \leq \text{Const.}(r\rho)^n$, and also $J_{n+3}(r'\rho) \leq \text{Const.}(r'\rho)^{n+3}$ when $n=1$. So the first integral resulting from that splitting is bounded. The integral over $[1,\infty[$ is also clearly bounded. Consider at last the second integral, and the most difficult case, i.e. $n=1$. This time, we estimate simply $J_{n+3}(r'\rho)$ by a constant, and write $J_n(r\rho) \leq \text{Const.}(r\rho)$; integrating with respect to ρ gives a term $\mathcal{O}(r\log r') = \mathcal{O}(\log r')$, but we can still divide this by r'^α, $0<\alpha<1$, since $r'\chi(r')r'^{-3+\alpha}$ is also integrable. All other terms can be handled similarly. So the Lemma is proved. ♣

Now we give some continuity properties of A_n on the space X_k^0.

Proposition a.2: Assume A_n has the asymptotic property and is positivity preserving. Then A_n is a bounded operator on X_k^0, $k = 1, 2$.

Proof: First we show that $r \mapsto A_n u(r)$ is continuous. Let χ be a smooth cutoff as above, we write $A_n u(r) = A_n(1-\chi)u(r) + A_n \chi u(r)$. As $A_n(r, r')$ is smooth and $(1-\chi)u$ of compact support, the first term is a continuous function of r. For the second term, write as in the proof of Lemma a.1 when $k = 2$,

$$(a.10) \qquad A_n \chi u(r) = \int_0^\infty dr' r' \frac{\chi(r')}{r'^3} \big(r'^2 u(r')\big) \int_0^\infty d\rho \mathcal{F}\widehat{J}(\rho) J_n(r\rho) r' \rho J_n(r'\rho)$$

and use(a.6) to show that the ρ-integral is a continuous function of (r, r'), uniformly bounded with respect to $r' \in [0, \infty[$. Since $u \in X_2^0$, $r'^2 u(r')$ is also bounded on $\operatorname{supp} \chi$, and $r' \frac{\chi(r')}{r'^3}$ integrable, we can conclude that $A_n u$ is continuous.

When $k = 1$, replace (a.10) by

$$A_n \chi u(r) = \int_0^\infty dr' r' \frac{\chi(r')}{r'^3} \big(r' u(r')\big) r' \int_0^\infty d\rho \mathcal{F}\widehat{J}(\rho) J_n(r\rho) r' \rho J_n(r'\rho)$$

and apply the procedure above. We shall have to estimate $\int_0^\infty \frac{d\rho}{\rho} J_n(r\rho) J_{n+2}(r'\rho) \widehat{J}(\rho)$, for r in a compact set we split again the integration over $[0, \infty[$into $[0, \rho_0]$ and $[\rho_0, \infty]$, using the bound $J_n(r\rho) \leq \text{Const.}(r\rho)^n$ for the first part, and absolute convergence for the second part. Thus we can conclude to continuity as in the case $k = 2$.

Next we show that if $u \in X_k^0$, then $r^k A_n u$ is bounded near infinity. Write $r^k A_n u(r) = r^k A_n(1-\chi)u(r) + r^k A_n \chi u(r)$, by (a.4) we estimate the first term by $\text{Const.}\|u; X_k^0\|$. For the second term, we have for all $r \geq 1$,$|u(r)| \leq \|u; X_k^0\| r^{-k}$, so using the fact that A_n has the asymptotic property and is positivity preserving, we get $|A_n u(r)| \leq \|u; X_k^0\| A_n(\cdot)^{-k}(r) \leq \text{Const.}\, r^{-k} \|u; X_k^0\|$. This brings the proof to an end. ♣

Along the same lines, we can prove continuity of A_n in the asymptotic space Y_k^0.

Proposition a.3: Assume A_n has the asymptotic property and is positivity preserving. Then A_n is a bounded operator on Y_k^0, $k = 1, 2$.

Moreover, if A_n has vanishing subprincipal symbol, then it preserves the closed subspace $E_0 = \{u \in X_k^0 : \ell_0(u) = 0\}$.

Proof: Consider first $k = 1$. We just need to show that $rA_n u(r)$ has a limit as $r \to \infty$ (the same as u,) and that

$$(a.12) \qquad \sup_{r \in [1, +\infty[} |r\big(rA_n u(r) - \ell(u)\big)| \leq C\|u; Y_1^0\|$$

For the first point, we observe that by definition, given $\varepsilon > 0$, $-\frac{\varepsilon}{r'} \leq u(r') - \frac{\ell(u)}{r'} \leq \frac{\varepsilon}{r'}$ for r' large enough. Multiply this inequality by $\chi(r')$ as before, use that A_n is positively preserving, and $A_n\big(\frac{\chi}{r'}\big)(r) \sim \frac{1}{r}$, we find $|A_n\chi u(r) - \frac{\ell(u)}{r}| \leq \frac{2\,\varepsilon}{r}$ for r large enough. On the other hand, by (a.4) for $k=2$, we have $A_n(1-\chi)u(r) = \mathcal{O}(r^{-2})$, $r \to \infty$, so we can conclude $\lim_{r\to\infty} rA_nu(r) = \ell(u)$. It is also clear that if A_n has vanishing subprincipal symbol, then it preserves the closed subspace $E_0 = \{u \in X_k^0 : \ell_0(u) = 0\}$. Now we prove (a.12). The same argument as before shows that if $u \in Y_1^0$, then $|A_n\chi u(r) - \ell(u)A_n\frac{\chi}{r'}(r)| \leq C\|u; Y_1^0\|A_n\frac{\chi}{r'^2}(r)$, so by the asymptotic property, for r large enough, $r^2|A_n\chi u(r) - \frac{\ell(u)}{r}| \leq C(\|u;Y_1^0\| + |\ell(u)|) \leq C'\|u;Y_1^0\|$. Also by (a.4), we have for $r \geq 1$, $r^2|A_n(1-\chi)u(r)| \leq C\sup_{r'\leq 1}|u(r')| \leq C\|u;Y_1^0\|$. This proves (a.12) and the continuity property for $k=1$.

The case $k=2$ goes similarly, taking advantage that (a.4) holds for $k = 5/2$. ♣

Our next result concerns the behavior of $A_nu(r)$ as $r \to 0$.

Proposition a.4: If if $u \in X_k^0$, then $r^{-n}A_n(r)$ is bounded near 0..

Proof: We use this time the relation

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{a.14}$$

which gives, by induction $2^{2p}(2p)x^{-2p}J_{2p}(x) = \sum_{j=-p}^{p}\alpha_jJ_{2(p+j)}(x)$, $n=2p$, and $2^{2p+1}(2p+1)x^{-2p-1}J_{2p+1}(x) = \sum_{j=-p}^{p+1}\beta_jJ_{2(p+j)}(x)$, $n = 2p+1$. where α_j and β_j are rational numbers. Then we argue as in Proposition a.2, for a linear combination of terms of the form

$$\int_0^\infty dr'r'u(r')\int_0^\infty d\rho\rho^{n+1}J_{2p+2j}(r\rho)J_n(r'\rho)\mathcal{F}\widehat{J}(\rho)$$

(we have assumed here $n = 2p+1$ the case $n = 2p$ is similar,)and split accoording to the partition of unity χ and $1-\chi$. The latter part is obviously bounded uniformly as $r \to 0$. For the χ-integrals, we divide by r'^3 as in the proof of Proposition a.2. This easily gives the Proposition. ♣

A more difficult problem would be to analyse the asymptotic behavior of $A_nu(r)$, as $r \to 0$. Formal analogy with radially symmetric solutions of the Ginzburg-Landau equations would suggest that if $u(r) \sim r^n$ as $r \to 0$, then so does $A_nu(r)$.

B. Renormalized free energy in infinite volume.

In this section we renormalize (c.31) in infinite volume on the real plane, restricting m to a class of functions having topological degree at infinity. As usual, the divergence is due to the self-energy corresponding to a neighborhood of the diagonal in $\int dxdyJ(x-y)|m(x)-m(y)|^2$, and to $\int dxf_\beta(m(x))$. For the latter, we shall remove $f_\beta(m_\beta)$ from $f_\beta(m(x))$ provided m stays sufficiently close to m_β. Our renormalized energy needs also to be bounded from below.

We discuss first some properties of the degree of a complex valued function. See [FoGa] for more advanced results. Let $m:\mathbf{R}^2\to\mathbf{C}$ be a differentiable function, considered as a vector field on $\mathbf{R}^2$, and subject to the condition $|m(x)|\to m_\beta$ as $|x|\to\infty$ uniformly in $\widehat{x}=x/|x|$. Then the integer

$$\deg_R m=\frac{1}{2\pi}\int_{|x|=R}d(\arg m)=\frac{1}{2\pi}\int_{|x|=R}\frac{dm}{m}\tag{b.1}$$

independent of R when $R>0$ is large enough, is called the (topological) degree of m at infinity, and denoted by $\deg_\infty m$. We do not attempt to characterize all functions satisfying (b.1), but just state for completeness the following result (the condition on the v_k could certainly be weakened.) We say that m have a one-sided Fourier series if there is $n\geq 0$ (the case $n\leq 0$ follows by complex conjugation,) such that $m(x)=v_n(r)e^{in\theta}+\sum_{k\geq n+1}v_k(r)e^{ik\theta}$.

Proposition b.1: Let m have a one-sided Fourier series, $v_n(r)\to m_\beta$ and $m(x)-v_n(r)e^{in\theta}\to 0$ as $r\to\infty$ uniformly in θ. Assume moreover the Fourier coefficients decay sufficiently fast to ensure existence of a continuous limit of $\sum_{k\geq n+1}v_k(r)z^k$ as $|z|\to 1^-$, i.e. $\sum_{k\geq n+1}k^2|v_k(r)|^2<\infty$, uniformly as $r\to\infty$. Then $\deg_\infty m=n$.

Proof: By assumption, we can differentiate the series term by term on the circle of center 0, and radius R. To compute $\frac{1}{2\pi}\int_{|x|=R}\frac{dm}{m}$, we put $z=e^{i\theta}$. Since $m(x)-v_n(r)e^{in\theta}\to 0$ as $r\to\infty$ uniformly in θ, $|\sum_{k\geq n+1}\frac{v_k(R)}{v_n(R)}z^{k-n}|<1$ on $|z|=1$ for R large enough, and Rouché theorem asserts that $v_n(R)+\sum_{k\geq n+1}v_k(R)z^{k-n}$ has no zeroes inside the unit disc. So the only pole inside the unit disc is $z=0$ and the corresponding residue is equal to n. ♣

Unless restricting to those m with $\deg_\infty m=0$, $\mathcal{F}(m|m^c)$ doesn't make sense in the limit $\Lambda\to\infty$, so we need a renormalization to remove the logarithmic singularity. More precisely we show how to renormalize $\mathcal{F}(m)$, with $\mathcal{F}(m)$ as in (2.1) and m of degree n. For simplicity we restrict to the case where m has a single Fourier mode, i.e. m belongs to the sector $\langle e_n(\theta)\rangle$. The idea is to remove in (2.1) a thin conic neighborhood of the diagonal $x=y$ in $\mathbf{R}^2$ from the double integral.

To give some flavor of the general argument, we forget about the free energy density of the mean field $f_\beta(|m|)$, and start with the particular case where $m(x)=m_\beta e^{in\theta}$, which belongs to E defined in (2.1) but of course, is not continuous at 0. Denote by $\mathcal{F}^0(m)=$

$\frac{1}{4}\int dxdyJ(x-y)|m(x)-m(y)|^2$ the interaction term. Let first $r_0>0$ and split

$$(b.9)\qquad \begin{aligned}\mathcal{F}^0(m)&=\frac{1}{4}\Big(\int_{|x|>r_0}dx\int_{|y|>r_0}dy\\&+\int_{|x|>r_0}dx\int_{|y|<r_0}dy+\int_{|x|<r_0}dx\int_{\mathbf{R}^2}dy\Big)J(x-y)|m(x)-m(y)|^2\end{aligned}$$

We are going to reduce (b.9) modulo integrable terms. Because J decays rapidly at infinity, it is easy to see that the last 2 terms are bounded. Consider then the first term, and for each x in the domain of integration, let $L_x=\{y\in\mathbf{R}^2:|y|>r_0,|ty+(1-t)x|>r_0,\forall t\in[0,1]\}$ denote the *light cone* issued from x, and $S_x=\{|y|>r_0\}\setminus L_x$ the corresponding *shadow cone.* Using the rapid decrease of J, we see that $\int_{|x|>2r_0}dx\int_{S_x}dy|m(x)-m(y)|^2$ is bounded. We compute $\frac{1}{4}\int_{|x|>r_0}dx\int_{|y|>r_0}dyJ(x-y)|m(x)-m(y)|^2$ by choosing polar coordinates $x=re^{i\theta}$, $x-y=\rho e^{i\varphi}$, the volume element is $|dx\wedge d\overline{x}\wedge dy\wedge d\overline{y}|=4\rho r|dr\wedge d\theta\wedge d\rho\wedge d\varphi|$, and now we perform the integration over $\Omega=\{\rho\geq 0,r\geq r_0,\varphi\in[0,2\pi],\theta\in[0,2\pi]\}$. Writing $y=r'e^{i\theta'}$ we have the relation

$$(b.10)\qquad r'\sin(\theta'-\varphi)=r\sin(\theta-\varphi)$$

Thus the first term in (b.9) rewrites as

(b.11)
$$\frac{1}{4}\int_{|x|>r_0}dx\int_{|x|>r_0}dyJ(x-y)|m(x)-m(y)|^2=2m_\beta^2\int_\Omega d\varphi\rho d\rho\widehat{J}(\rho)rdrd\theta\big(1-\cos n(\theta-\theta')\big)$$

We make the following observations : for fixed φ, let Γ_φ be the reflection on the line $\theta=\varphi$. Then for all $r>\rho$, the map $(r,\theta)\mapsto(r',\theta')$, is 2π-periodic in θ (it corresponds to shifting the circle of center 0 and radius r by the vector $\rho(\cos\varphi,-\sin\varphi)$.) For given r, the function $\theta\mapsto r'$ is even under Γ_φ, and increases from $r-\rho$ for $\theta=\varphi$, to $r+\rho$ for $\theta=\varphi+\pi$, while $\theta\mapsto\theta'-\theta$ is odd under Γ_φ, increases from 0 for $\theta=\varphi$ to a maximum value $(\theta'-\theta)_{\max}$, with $\cos((\theta'-\theta)_{\max})=r(r^2+\rho^2)^{-1/2}$, for $\theta=\varphi+\pi/2$, and decreases again to 0 for $\theta=\varphi+\pi$. Together with relation (b.10) this gives

$$(b.12)\qquad |r'-r|/r\leq\rho/r,\quad|(\theta'-\theta)_{\max}|\approx\rho/r$$

We compute (b.11) as follows. For $C>0$ large enough to be fixed later, we split the integral over the sets $\{r\geq C\rho\}$, and $\{r\leq C\rho\}$. For the first part, $(\theta'-\theta)_{\max}$ is a (non degenerate) critical point, and we observe that for $C>0$ large enough :

$$(b.13)\quad 2\int_0^{2\pi}d\theta\big(1-\cos n(\theta-\theta')\big)=8\int_\varphi^{\varphi+\pi/2}d\theta\big(1-\cos n(\theta-\theta')\big)=\pi n^2\big(\frac{\rho}{r}\big)^2+\mathcal{O}\big(\frac{\rho}{r}\big)^3$$

where $\mathcal{O}\big(\frac{\rho}{r}\big)^3$ is asymptotic, as $\rho/r \to 0$, to $c_n\big(\frac{\rho}{r}\big)^3$, for some $c_n > 0$, all n (c_n increases to $+\infty$ with n.) This shows that the corresponding integral contributes with a logarithmic singularity

$$(b.14) \qquad \pi n^2 \int_{\Omega_0} d\rho\rho^3 \widehat{J}(\rho)\frac{dr}{r}, \quad \Omega_0 = \{r > r_0, r \geq C\rho\}$$

which we substract from $2\int_\Omega d\theta d\varphi r dr \rho d\rho \widehat{J}(\rho)\big(1-\cos n(\theta-\theta')\big)$. [Actually we replace the integration over $r > r_0$ in (b.13) and (b.14) by $N > r > r_0$ and let then $N \to \infty$.] Since the θ-integral is clearly independent of φ, integrating over φ gives an additional factor of 2π. The term $\mathcal{O}\big(\frac{\rho}{r}\big)^3$ in (b.13) then contributes to a finite, and positive integral. Consider next the integral over $r \leq C\rho$. Interchanging the $drd\rho$ integrals (which is legitimate after the cutoff $r < N$,) we are lead to estimate

$$2\int_0^\infty rdr \int_{r/C}^\infty d\rho\rho\widehat{J}(\rho)\int_0^{2\pi} d\varphi \int_0^{2\pi} d\theta\big(1-\cos n(\theta-\theta')\big)$$

There we use simply that the $d\varphi d\theta$ integral is bounded and take advantage of the rapid decrease of J to bound the ρ-integral by a negative power of r to make convergent the resulting r-integral. Thus we proved

Lemma b.4: With the notations above, if $m(x) = m_\beta e^{in\theta}$, let

$$(b.15) \qquad \begin{aligned} &\mathrm{ren}\Big(\frac{1}{4}\int dx \int dy J(x-y)|m(x)-m(y)|^2\Big) \\ &= \frac{1}{4}\int dx \int dy J(x-y)|m(x)-m(y)|^2 - 2(\pi n m_\beta)^2 \int_{\Omega_1} d\rho\rho^3\widehat{J}(\rho)\frac{dr}{r} \end{aligned}$$

Then $0 < c_n \leq \mathrm{ren}\big(\frac{1}{4}\int dx \int dy J(x-y)|m(x)-m(y)|^2\big) < +\infty$.

Now we turn to the more general case, which is sufficient for our purposes, where $m(x)$ belongs to the $\langle e_n\rangle$-sector, and $|m(x)| \to m_\beta$ as $x \to \infty$. First we renormalize the mean field free energy $\mathcal{F}^1(m) = \int dx f_\beta(m(x))$ for $|m| - m_\beta \in L^2(\mathbf{R}^2)$; taking advantage of the fact that f_β attains its minimum at m_β, we let simply

$$(b.16) \qquad 0 \leq \mathcal{F}^1_{\mathrm{ren}}(m) = \int dx\big(f_\beta(m(x)) - f_\beta(m_\beta)\big) < +\infty$$

he first inequality because . Next we pass to the interaction term $\mathcal{F}^0(m)$ as we did for $m(x) = m_\beta e^{in\theta}$. So let $r_0 > 0$ and split the integral as in (b.9) ; as before we are left with the integral over $\{|x| > r_0, |y| > r_0\}$. Assume also that m is continuously differentiable there. Use Taylor formula to rewrite $m(x) - m(y) = \int_0^1 \nabla m(ty + (1-t)x)dt \cdot (x-y)$, let

$z = ty+(1-t)x = re^{i\theta}, x-y = \rho e^{i\varphi}$. Expanding the product, multiplying by $J(x-y) = \widehat{J}(\rho)$, computing the Jacobian, and integrating, we find

$$
\begin{aligned}
&\frac{1}{4}\int_{|x|>r_0} dx \int_{L_x} dy J(x-y)|m(x)-m(y)|^2 = \int_0^1 dt \int_0^1 dt' \int_0^\infty \widehat{J}(\rho)\rho^3 d\rho \int_0^{2\pi} d\varphi \\
&\quad \times \int_0^{2\pi} d\theta \int_{r_0}^\infty r dr \big[\frac{\partial m}{\partial r}(z)\frac{\partial \overline{m}}{\partial r'}(z')\cos(\varphi-\theta)\cos(\varphi-\theta') \\
(b.20) \qquad &\quad + \frac{1}{r}\frac{\partial m}{\partial \theta}(z)\frac{\partial \overline{m}}{\partial r'}(z')\sin(\varphi-\theta)\cos(\varphi-\theta') + \frac{\partial m}{\partial r}(z)\frac{1}{r'}\frac{\partial \overline{m}}{\partial \theta'}(z')\sin(\varphi-\theta')\cos(\varphi-\theta) \\
&\quad + \frac{1}{r}\frac{\partial m}{\partial \theta}(z)\frac{1}{r'}\frac{\partial \overline{m}}{\partial \theta'}(z')\sin(\varphi-\theta)\sin(\varphi-\theta')\big]
\end{aligned}
$$

where $z' = t'y+(1-t')x = r'e^{i\theta'}$. Because $y \in L_x$, the segment $[x,y]$ lies outside $B_2(0,r_0)$ so $r, r' \geq r_0$ and all terms in the integral on the RHS of (b.20) are well defined. To start with, we make as before a cut-off in the r-integral in the RHS, so that we can split it into different terms. The domain of integration is denoted by $\widetilde{\Omega} = \{0 \leq t, t' \leq 1, r > r_0, \rho > 0, \varphi, \theta \in [0, 2\pi]\}$. The observations leading to (b.12) can be exactly repeated, changing x to z, y to z', and ρ to $|t-t'|\rho$, now we get :

$$
(b.21) \quad |r'-r|/r \leq |t-t'|\rho/r,\ \cos((\theta'-\theta)_{\max}) = r(r^2+(t-t')^2\rho^2)^{-1/2},\ |(\theta'-\theta)_{\max}| \approx |t-t'|\rho/r
$$

Making use of the symmetry in θ, θ', (b.20) is the sum of 3 integrals, whose corresponding integrands write :

$$
\begin{aligned}
&A(t,t',\rho,r,\varphi,\theta) = u'(r)u'(r')\cos(\varphi-\theta)\cos(\varphi-\theta')\cos n(\theta-\theta') \\
&B(t,t',\rho,r,\varphi,\theta) = 2n\big[\frac{u(r)}{r}u'(r')\sin(\varphi-\theta)\cos(\varphi-\theta') \\
(b.23) \qquad &\qquad\qquad + u'(r)\frac{u(r')}{r'}\sin(\varphi-\theta')\cos(\varphi-\theta)\big]\sin n(\theta-\theta') \\
&D(t,t',\rho,r,\varphi,\theta) = n^2\frac{u(r)u(r')}{rr'}\sin(\varphi-\theta)\sin(\varphi-\theta')\cos n(\theta-\theta')
\end{aligned}
$$

($u'(r)$ denotes the r-derivative of u.) We examine first D, which we renormalize by setting $u(r) = m_\beta + v(r)$; using (b.10) we find :

$$
(b.24) \qquad D = n^2\big[\big(\frac{m_\beta}{r}\big)^2 + \frac{m_\beta}{r^2}(v(r)+v(r')) + r^{-2}v(r)v(r')\big]\sin^2(\varphi-\theta)\cos n(\theta-\theta')
$$

Comparing the coefficients of m_β^2 in (b.20) and (b.11), (b.13) shows that we must have, for $r \geq C\rho$:

$$
(b.25) \qquad n^2\big(\frac{\rho}{r}\big)^2\int_0^1 dt\int_0^1 dt'\int_0^{2\pi} d\theta \sin^2(\varphi-\theta)\cos n(\theta-\theta') = \pi n^2\big(\frac{\rho}{r}\big)^2 + \mathcal{O}\big(\frac{\rho}{r}\big)^3
$$

Since the θ-integral is clearly independent of φ, integrating over φ gives again an additional factor of 2π, so that $\int_{\widetilde{\Omega}} dtdt'd\varphi\widehat{J}(\rho)\rho^3 d\rho r dr n^2\big(\frac{m_\beta}{r}\big)^2 \sin^2(\varphi-\theta)\cos n(\theta-\theta')$ contributes to (b.20) [after letting $N\to\infty$] with the logarithmic singularity

$$2(\pi n m_\beta)^2\int_{\Omega_0} d\rho\rho^3\widehat{J}(\rho)\frac{dr}{r},\quad \Omega_0=\{r>r_0, r\geq C\rho\}$$

which we eventually substract from (b.20), and the term $\mathcal{O}\big(\frac{\rho}{r}\big)^3$ in (b.25) then contributes with a finite, and positive integral.

Consider next the integral over $r_0<r\leq C\rho$. Interchanging the $drd\rho$ integrals we are lead to estimate

$$\int_{r_0}^{\infty} drr\big(\frac{m_\beta}{r}\big)^2\int_{r/C}^{\infty} d\rho\rho^3\widehat{J}(\rho)\int_0^{2\pi} d\varphi\int_0^{2\pi} d\theta\sin^2(\varphi-\theta)\cos n(\theta-\theta')$$

There again, we take advantage of the rapid decrease of $\widehat{J}$ to bound the ρ-integral by a negative power of r to make convergent the resulting r-integral. Now we examine all other terms contributing to (b.20), namely $A(t,t',\rho,r,\varphi,\theta)$, $B(t,t',\rho,r,\varphi,\theta)$, and the part

$$D'(t,t',\rho,r,\varphi,\theta)=n^2\big[\frac{m_\beta}{r^2}(v(r)+v(r'))+r^{-2}v(r)v(r')\big]\sin^2(\varphi-\theta)\cos n(\theta-\theta')$$

which is left from D. Consider first D', and recall $v\in L^2(\mathbf{R}^+;rdr)$. For the first term we use Cauchy-Schwarz inequality to get

$$|\int_{r_0}^{\infty} rdr\frac{m_\beta}{r^2}v(r)\sin^2(\varphi-\theta)\cos n(\theta-\theta')|\leq\big(\int_{r_0}^{\infty} rdr|v(r)|^2\big)^{1/2}\big(\int_{r_0}^{\infty} rdr\big(\frac{m_\beta}{r^2}\big)^2\big)^{1/2}$$

and similarly for the second term, if we think of the fact that r and r' play symmetric rôles. Integrating against $\widehat{J}(\rho)\rho^3$ with respect to $dtdt'd\varphi d\theta d\rho$ gives finite quantities. The last term in D' contains the correlations $r^{-2}v(r)v(r')$. Again we split the integration according to $\{r\geq C\rho\}$ and $\{r\leq C\rho\}$, and for the second part, use the rapid decrease of $\widehat{J}$. For the first part, we use instead that the "translations" $(t,t',\varphi,\theta,\rho)\mapsto\big(r\mapsto v(r')\big)$ are uniformly continuous in $L^2([C\rho,+\infty[;rdr)$ when $C\rho\leq r_0$; this gives

$$\begin{aligned}|\int_{\widetilde{\Omega},r>C\rho} & dtdt'd\varphi\widehat{J}(\rho)\rho^3 d\rho rdrd\theta r^{-2}v(r)v(r')\\ &\times\sin^2(\varphi-\theta)\cos n(\theta-\theta')|\leq\text{Const.}\,\|v;L^2([r_0,+\infty[;rdr)\|^2\end{aligned}$$

Consider then A or B. These terms involve the derivatives $r^{-1}v'(r')$, $r'^{-1}v'(r)$, and the correlations $v'(r)v'(r')$, $r^{-1}u(r)v'(r')$, $r'^{-1}u(r')v'(r)$, so they can be treated as above, provided $r^\delta v'(r)$ is in $L^2(rdr)$ for some $\delta>0$. This holds true when $u(r)\sim m_\beta+\mathcal{O}(1/r)$, $r\to\infty$ and this relation can be differentiated, e.g. if m has the asymptotic properties of a symbol in $1/r$. Summing up, we proved the :

Proposition b.5: If $m(x) = u(r)e^{in\theta}$ is bounded, $u = m_\beta + v$, with

$$v \in \mathcal{W} = \{L^2([0,1]; r^{-1}dr) \cap H^1(\mathbf{R}^+; rdr), (\cdot)^\delta v' \in L^2([1, +\infty[; rdr)\}$$

for some $\delta > 0$, then :

$$\begin{aligned} \mathcal{F}_{\mathrm{ren},r_0}(m) =& \frac{1}{4}\int dx \int dy J(x-y)|m(x)-m(y)|^2 - 2(\pi n m_\beta)^2 \int_{\Omega_0} d\rho\rho^3 \widehat{J}(\rho)\frac{dr}{r} \\ &+ \int dx\big(f_\beta(m(x)) - f_\beta(m_\beta)\big) < \infty \end{aligned} \tag{b.28}$$

Moreover, $\mathcal{F}_{\mathrm{ren}}$ is (strongly) continuous on $\mathcal{W}$ endowed with its natural Hilbert space structure.

Remark b.6: The renormalization can be easily extended to the case where u has a discontinuity on a sphere $r = \lambda$, i.e. when it has derivatives a.e. This is used when considering the partial dynamics.

When m is bounded, it is clear that $\mathcal{F}^0_{\mathrm{ren},r_0}(m)$ and $\mathcal{F}^0_{\mathrm{ren},r_1}(m)$ will just differ by a quantity of order $(r_0 - r_1)^4$. Next we show that if $J \geq 0$, then $\mathcal{F}_{\mathrm{ren},r_0}(m)$ is bounded from below in some region of the configuration space. First we consider the interaction term $\mathcal{F}^0(m) = \frac{1}{4}\int dxdy J(x-y)|m(x)-m(y)|^2$, and compute the formal Fourier transform of $h : \mathbf{R}^2 \to \mathbf{C}$, $h(x,y) = \sqrt{J(x-y)}\big(m(x)-m(y)\big)$. We have $\mathcal{F}h(\xi,\eta) = \big(\mathcal{F}\sqrt{J}(-\eta) - \mathcal{F}\sqrt{J}(\xi)\big)\mathcal{F}m(\xi+\eta)$, so by Parseval identity (still in the formal sense)

$$\int dxdy J(x-y)|m(x)-m(y)|^2 = (2\pi)^{-4}\int d\xi d\eta |\mathcal{F}m(\xi-\eta)|^2\big(\mathcal{F}\sqrt{J}(\eta) - \mathcal{F}\sqrt{J}(\xi)\big)^2$$

so we have exchanged the rôles of m and J. Of course the integral is divergent because $m \notin L^2$, but the singularity is due to $|\mathcal{F}m(\xi-\eta)|^2$, not to the boundary condition at infinity, since $\deg_\infty \mathcal{F}\sqrt{J} = 0$. The finite part (P.f.) of $|\mathcal{F}m(\xi-\eta)|^2$ is $|\mathcal{F}m(\xi-\eta)|^2|_{\xi=\eta} = \big(\int m dx\big)^2$ and is finite when $m(x) = e^{in\theta}u(r)$, $u(r) = m_\beta + v(r)$, $v \in L^2$, because of the periodicity in θ. So another renormalization of $\mathcal{F}^0(m)$ is given by

$$\mathcal{F}^0_{\mathrm{ren},\mathcal{F}}(m) = \frac{1}{4}(2\pi)^{-4}\ \mathrm{P.f.}\int d\xi d\eta |\mathcal{F}m(\xi-\eta)|^2\big(\mathcal{F}\sqrt{J}(\eta) - \mathcal{F}\sqrt{J}(\xi)\big)^2$$

which is obviously positive. Moreover, Lemma b.4 and its proof show that both renormalizations should agree up to a finite term, depending only on r_0, when $m(x) = m_\beta e^{in\theta}$. Thus, we should have $\mathcal{F}^0_{\mathrm{ren},r_0}(m) \equiv \mathcal{F}^0_{\mathrm{ren},\mathcal{F}}(m)$ modulo a constant term, depending on r_0, but not on

m, when m satisfies all hypotheses of Proposition b.5. In particular, $\mathcal{F}^0_{\mathrm{ren},r_0}(m)$ is bounded from below.

C). The free energy in Kac's model with continuous symmetry.

In this Section, we recall and make more precise the procedure of renormalization in the continuum, carried out in [AlBeCaPr], [Pr] for spins valued in $\{-1,+1\}$, and extended to the XY model in [BuPi], when a vorticity condition holds at infinity and the interaction is not necessarily compactly supported. Actually, the final form for the continuous renormalized free energy functional should be regarded as a postulate.

Note that passing from the lattice to the continuum amounts to consider convergence of Riemann sums as the mesh of the (scaled) lattice goes to 0, as an homogenization process, i.e. the convergence of discrete measures.

a) Some definitions.

Consider the lattice $\mathbf{Z}^d$, consisting in a bounded, connected domain $\widetilde{\Lambda}$ (the interior region), and its complement (the exterior region) $\widetilde{\Lambda}^c$ (twiddled letters will always denote discrete objects on $\mathbf{Z}^d$.) Physical systems make sense in the thermodynamical limit $\widetilde{\Lambda} \to \mathbf{Z}^d$, in the sense of Fisher. The simplest way of taking this limit is to double the side of the unit hypercube repeatedly, so the side of $\widetilde{\Lambda} = \widetilde{\Lambda}_{\widetilde{\ell}}$ is of the form $2^{\widetilde{\ell}}, \widetilde{\ell} \in \mathbf{N}$.

To each site $i \in \mathbf{Z}^d$ is attached a classical spin variable $\sigma(i) \in \mathbf{S}^q$. The configuration space $\mathcal{X}(\mathbf{Z}^d) = (\mathbf{S}^{q-1})^{\mathbf{Z}^d}$ is the set of all such classical states of spin ; it has the natural internal symmetry group $O^+(q)$ ($q = 2$ for the planar rotator.) Given the partition $\mathbf{Z}^d = \widetilde{\Lambda} \cup \widetilde{\Lambda}^c$, we define by restriction the interior and exterior configuration spaces $\mathcal{X}(\widetilde{\Lambda})$ and $\mathcal{X}(\widetilde{\Lambda}^c)$, and the restricted configurations by $\sigma_{\widetilde{\Lambda}}$ and $\sigma_{\widetilde{\Lambda}^c}$.

It is convenient to rescale $\widetilde{\Lambda}_\ell$ to a domain of fixed size $L = 2^\ell$, $\ell \in \mathbf{N}$, $\Lambda \subset \mathbf{R}^d$, with scaling factor is $2^{-\widetilde{\ell}}$; for the moment we could think of Λ as the unit square ($\ell = 0$) but we shall eventually let also $\Lambda \to \infty$.

Following [Pr], for $k \in \mathbf{N}$, we denote by $\mathcal{Q}^{(k)}$ the partition of $\mathbf{R}^d$ into small cubes $C^{(k)} = \{r = (r_1, \cdots, r_d) \in \mathbf{R}^d, 2^{-k}x_i \leq r_i < 2^{-k}(x_i + 1)\}$, of side 2^{-k}, and indexed by $x = (x_1, \cdots, x_d) \in \mathbf{Z}^d$, called ($\mathcal{Q}^{(k)}$-)atoms. The atom $C^{(k)}(r)$ is the unique atom of $\mathcal{Q}^{(k)}$ that contains r. We say also that a function on $\mathbf{R}^d$ is $\mathcal{Q}^{(k)}$-measurable if it is constant on each atom of $\mathcal{Q}^{(k)}$, and a set $A \subset \mathbf{R}^d$ is $\mathcal{Q}^{(k)}$-measurable if its indicator function is $\mathcal{Q}^{(k)}$-measurable. This allows in a natural way to identify a function σ on the lattice with a $\mathcal{Q}^{(k)}$-measurable function $\sigma^{(k)}$ on $\mathbf{R}^d$, assuming $\sigma^{(k)}(r) = \sigma(x)$ with $x = (x_1, \cdots, x_d)$ and $r = (r_1, \cdots, r_d)$ as above.

Let now γ of the form $\gamma = 2^{-k_\gamma}$, $k_\gamma \in \mathbf{N}$, that will be the inverse of the interaction length in Kac's potential. Given a state $\sigma \in \mathcal{X}(\mathbf{Z}^d)$, we define σ_γ as the $\mathcal{Q}^{(k_\gamma)}$-measurable function

$\sigma^{(k_\gamma)}$. Since we take a simultaneous limit $\widetilde{\Lambda}\to\infty$, $\gamma\to 0$, rather than Lebowitz-Penrose limit $\widetilde{\Lambda}\to\infty$ followed by $\gamma\to 0$, it may be convenient to label the configurations by γ, instead of Λ. We call also σ_γ a *smooth-grained* configuration.

Because of the internal continuous symmetry of $\mathcal{X}(\Lambda)$, the probability distribution ν for the states of spin is defined as the normalized surface measure on $\mathbf{S}^{q-1}$, i.e. $\nu(d\sigma_i)=\omega_q^{-1}\delta(|\sigma_i|-1)d\sigma_i$, where ω_q is the volume of $\mathbf{S}^{q-1}$.

Given σ_γ as above, and an integer $n_\gamma\leq k_\gamma$, we associate the $\mathcal{Q}^{(n_\gamma)}$-measurable function (magnetization)

$$(c.1)\qquad m_\gamma(r)=\pi^{(n_\gamma)}\sigma_\gamma(r)=\frac{1}{|C^{(n_\gamma)}|}\int_{C^{(n_\gamma)}(r)}dr'\sigma_\gamma(r')$$

These averages of σ_γ over the "intermediate" boxes (or block spins) $C^{(n_\gamma)}(r)$ of volume $\gamma^{\delta d}$, define the *coarse-grained* configurations, and the map $\pi^{(n_\gamma)}:\Omega\to B'_q(0,1)$, $\Omega=\big(\mathbf{S}^{q-1}\big)^{\widetilde{C}^{(n_\gamma)}}$, $B'_q(0,1)$ the closed unit ball of $\mathbf{R}^q$, is called the *block-spin transformation.* We may think of it as a random variable.

More generally, let $\pi_N:\big(\mathbf{S}^{q-1}\big)^N\to B'_q(0,1),\sigma=(\sigma_1,\cdots,\sigma_N)\mapsto\pi_N(\sigma)=\frac{1}{N}\sum_{i=1}^N\sigma_i$. It is easy to see that π_N is a smooth, surjective map, and its restriction $\widetilde{\pi}_N:\big(\mathbf{S}^{q-1}\big)^N\setminus\Delta\to B_q(0,1)$ a submersion, where Δ denotes the diagonal $\sigma_1=\cdots=\sigma_N$ of $\big(\mathbf{S}^{q-1}\big)^N$, and $B_q(0,1)$ the open unit ball of $\mathbf{R}^q$. Hence the probability measure $\nu_N=(\pi_N)_*(\nu\otimes\cdots\otimes\nu)$ has a smooth density with respect to the (normalized) Lebesgue measure on $B_q(0,1)$, namely $\frac{d\nu_N}{dm}(m)=\int_{\left(\mathbf{S}^{q-1}\right)^N}\prod_{i=1}^N\nu(d\sigma(i))\delta\big(\pi_N(\sigma)-m\big)$, and we can check as in [BuPi,formula (5.1)] that

$$\frac{d\nu_N}{dm}(m)=\big(\frac{N}{2\pi}\big)^q\int_{\mathbf{R}^q}dve^{-iN\langle v,m\rangle}\big(\int_{\mathbf{S}^{q-1}}\nu(dx)e^{i\langle v,x\rangle}\big)^N$$

(here $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbf{R}^q$.)

Of course, Kolmogorov realization theorem would allow to define this way a probability measure on $\big(B_q(0,1)\big)^\Lambda$, but this will not be used at the present level.

Now, following [Pr] we choose n_γ so that $\gamma^\delta=2^{-n_\gamma}$, for some $0<\delta<1$, a good choice is $\delta=1/2$. From the point of view of the discrete scheme, a coarse configuration is defined on a "coarse lattice" $\widetilde{\Lambda}^*$ which we magnify by the factor $2^{k_\gamma-n_\gamma}$ to the "smooth lattice" $\widetilde{\Lambda}$ (see e.g. [El-BoRo].) Thermodynamical properties of the system are most significant on the coarse lattice.

From the discussion above, for $k\in\mathbf{N}$, let $\widetilde{C}^{(k)}=2^kC^{(k)}\cap\mathbf{Z}^d$ denote the atom $C^{(k)}$ rescaled to the lattice units ; thus, $\widetilde{C}^{(k)}$ is the rescaled block-spin. So the probability distribution $\nu^{(n_\gamma)}$ of the empirical average $\pi^{(n_\gamma)}$ has Radon-Nikodym density

$$(c.2)\qquad \frac{d\nu^{(n_\gamma)}}{dm}(m) = \int_\Omega \prod_{i\in\widetilde{C}^{(n_\gamma)}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)\delta\big(\pi^{(n_\gamma)}\sigma_{\widetilde{\Lambda}}(i) - m\big), \quad \Omega = \big(\mathbf{S}^{q-1}\big)^{\widetilde{C}^{(n_\gamma)}}$$

We shall sometimes write $\sigma_\gamma(i)$ instead of $\sigma_{\widetilde{\Lambda}}(i)$. Let also $N = |\widetilde{C}^{(n_\gamma)}| = \gamma^{(\delta-1)d}$, $\nu^{(n_\gamma)} = \nu_N$.

The continuous Kac Hamiltonian is defined as follows. Let $J \geq 0$ be the interaction potential, for a given $\sigma^c \in \mathcal{X}(\widetilde{\Lambda}^c)$, and for $\Lambda = \bigcup C^{(k_\gamma)} = \bigcup C^{(n_\gamma)}$ as above, we define the energy of a configuration $\sigma \in \mathcal{X}(\widetilde{\Lambda})$ by

(c.3)

$$H(\sigma_\gamma|\sigma_\gamma^c) = -\frac{1}{2}\int_\Lambda dr \int_\Lambda dr' J(r-r')\langle\sigma_\gamma(r),\sigma_\gamma(r')\rangle - \int_\Lambda dr \int_{\Lambda^c} dr' J(r-r')\langle\sigma_\gamma(r),\sigma_\gamma^c(r')\rangle$$

Again from the point of view of the discrete scheme, since $\sigma_\gamma(\gamma i) = \sigma(i)$, for $i \in \mathbf{Z}^d$, there is a corresponding Hamiltonian on $\mathbf{Z}^d$ defined by

$$(c.4)\qquad \widetilde{H}_\gamma(\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c}) = -\frac{1}{2}\sum_{i,j\in\widetilde{\Lambda}} J_\gamma(i,j)\langle\sigma(i),\sigma(j)\rangle - \sum_{(i,j)\in\widetilde{\Lambda}\times\widetilde{\Lambda}^c} J_\gamma(i,j)\langle\sigma(i),\sigma(j)\rangle$$

where

$$(c.5)\qquad J_\gamma(i,j) = \gamma^{-d}\int_{C^{(n_\gamma)}(\gamma i)} dr \int_{C^{(n_\gamma)}(\gamma j)} dr' J(r-r')$$

The two hamiltonians are simply related by :

$$(c.6)\qquad \widetilde{H}_\gamma(\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c}) = \gamma^{-d}H(\sigma_\gamma|\sigma_\gamma^c)$$

so that H is an intensive hamiltonian. As observed in [AlBeCaPr], neglecting the variations of J in the integral, we get

$$(c.7)\qquad J_\gamma(i,j) \approx \gamma^d J(\gamma|i-j|)$$

which has the typical scaling properties of the original Kac potential, and the results of this Section remain valid when the energy is given by (c.4) with (c.7) holding as an equality. This observation allows to define Gibbs measure conditioned by $\sigma_{\widetilde{\Lambda}^c}$, on the space of spin configurations $\mathcal{X}(\widetilde{\Lambda})$ with mesh γ, at inverse temperature β, as

$$(c.8)\qquad \mu_{\beta,\gamma,\widetilde{\Lambda}}(d\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c}) = \frac{1}{Z_{\beta,\gamma,\widetilde{\Lambda}}(\sigma_{\widetilde{\Lambda}^c})}\exp\big[-\beta\widetilde{H}_\gamma(\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c})\big]\prod_{i\in\widetilde{\Lambda}}\nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)$$

where

$$(c.9) \qquad Z_{\beta,\gamma,\widetilde{\Lambda}}(\sigma_{\widetilde{\Lambda}^c}) = \int_{\Omega_0} \prod_{i\in\widetilde{\Lambda}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big) \exp\big[-\beta \widetilde{H}_\gamma(\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c})\big], \quad \Omega_0 = \big(\mathbf{S}^{q-1}\big)^{\widetilde{\Lambda}}$$

is the partition function, making of $\mu_{\beta,\gamma,\widetilde{\Lambda}}(\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c})$ a probability measure on the product space. As before, we can take the direct image $\mu_{\beta,\gamma,\widetilde{\Lambda}}(d\sigma_{\widetilde{\Lambda}}|\sigma_{\widetilde{\Lambda}^c})$ through the block-spin transformation. We shall discuss this in the following subsections.

b) Entropy estimates.

We want to relate $\frac{1}{N}\log\frac{d\nu_N}{dm}(m)$ with the entropy functional $I(m)$ of the mean field approximation. Recall $I(m) = \sup_{k\in\mathbf{R}^q}\big(\langle k,m\rangle - \log\phi(k)\big)$, $\phi(k) = \int_{\mathbf{S}^{q-1}} e^{\langle k,v\rangle}\nu(dv)$, and $I(m) = \langle k^*,m\rangle - \log\phi(k^*)$ where $k^* = k^*(m)$ is the unique point in $\mathbf{R}^q$ that achieves the maximum. Clearly also, by spherical symmetry, $I(m) = \widehat{I}(|m|)$, and $\widehat{I}(|m|) = \sup_{t\geq 0}\big(t|m| - \log\widehat{\phi}(t)\big) = t^*|m| - \log\widehat{\phi}(t^*)$, where $t^* = t^*(|m|)$, $t^*(0) = 0, t^*(\rho) \sim (2-2\rho)^{-1}$ as $\rho\to 1$. Furthermore the supremum is achieved when k^* and m are colinear. For $|m|<1$ we introduce the probability measure $\mu(dx;m)$ on $\mathbf{S}^{q-1}$ defined by

$$(c.11) \qquad \mu(dx;m) = \exp(\langle k^*,x\rangle - \log\phi(k^*))\nu(dx)$$

As in (c.2) we define

$$\frac{d\mu^{(n_\gamma)}}{dm}(m) = \int_\Omega \prod_{i\in\widetilde{C}^{(n_\gamma)}} \mu\big(d\sigma_{\widetilde{\Lambda}}(i);m\big)\delta\big(\pi^{(n_\gamma)}\sigma_{\widetilde{\Lambda}}(i) - m\big)$$

and denote $\mu_N(dx;m) = \mu^{(n_\gamma)}(dx;m)$. It is easy to see that $\frac{d\nu_N}{dm}(m) = e^{-NI(m)}\frac{d\mu_N}{dm}(m)$. Let also φ_m be the complex function defined on $\mathbf{R}^q$

$$\varphi_m(v) = e^{i\langle v,m\rangle}\int_{\mathbf{S}^{q-1}} \mu(dx;m)e^{-i\langle v,x\rangle}$$

Recall from [BuPi,formula(5.7)] the identity

$$(c.12) \qquad \frac{1}{N}\log\frac{d\nu_N}{dm}(m) + I(m) = \frac{1}{N}\log\big[\big(\frac{N}{2\pi}\big)^q\int_{\mathbf{R}^q} dv\varphi_m(v)^N\big]$$

The observation is that $\frac{1}{N}\log\frac{d\nu_N}{dm}(m)$ is a small correction to $-I(m)$ (the entropy for the mean field) as N becomes large. Indeed we have :

Lemma c.1: Let $q=2$ for simplicity. With the notations above, we have for all $N\geq 1$,

$$(c.13) \qquad |\frac{1}{N}\log\frac{d\nu_N}{dm}(m) + I(m)| \leq \frac{1}{N}\log[C_0\big(\frac{N}{2\pi}\big)^q(N^q + N^{q'}(1-|m|)^{-1/2})]$$

for some $C_0 > 0, q' \geq 0$.

Proof: We need to show that the integral on the RHS of (c.12) grows at most linearly in $(1-|m|)^{-1/2}$, with coefficients polynomial in N. Notice first that $|\varphi_m(v)| \leq \varphi_m(0) = 1$ for all $v \in \mathbf{R}^q$, so integrating $\varphi_m(v)^N$ over the ball in $\mathbf{R}^q$ of center 0 and radius N we get

$$(c.15) \qquad |\int_{|v|\leq N} dv \varphi_m(v)^N| \leq \text{Const.}\, N^q$$

Now, we estimate the integral near ∞, using complex stationary phase. We will be a little sketchy, but it is easy to see that our leading terms give the correct behavior with the required uniformities (see e.g. [Sj] for more details.) Let $v = r(\cos\varphi, \sin\varphi)$, $\varphi \in [-\pi, \pi]$, we rewrite $\varphi_m(v)$ as

$$(c.16) \qquad \varphi_m(v) = \big(2\pi\widehat{\phi}(t^*)\big)^{-1} e^{ir|m|\sin\varphi} \int_{-\pi}^{\pi} d\theta e^{-ir\Phi(\theta,\varphi)}, \quad \widehat{\phi}(t^*) = I_0(t^*)$$

with $\Phi(\theta,\varphi) = \cos(\theta-\varphi) + i\lambda\sin\theta$, and $\lambda = t^*/r$. Here we consider $r \geq N$ as the large parameter. The critical points in θ are given by the equation $\sin(\theta-\varphi) - i\lambda\cos\theta = 0$, so $\theta \mapsto \Phi(\theta,\varphi)$ has no real critical point if $\varphi \neq \pm\pi/2$, and 2 real critical points $\theta = \pm\varphi$ otherwise. Actually, $\operatorname{Im}\Phi(\pm\pi/2, \mp\pi/2) < 0$ so the contribution of the critical point with sign opposite to this of φ will be exponentially small, and by symmetry it suffices to consider $(\theta,\varphi) = (\pi/2, \pi/2)$. This is a non degenerate critical point, since $\frac{\partial^2\Phi}{\partial\theta^2}(\pi/2,\pi/2) = -1 - i\lambda$. Because of analyticity, there is a complex critical point $\theta_c = \theta_c(\varphi)$ for nearby values of φ, and a simple calculation yields

$$(c.17) \qquad \Phi(\theta_c,\varphi) = 1 + i\lambda - \frac{i\lambda}{2(1+i\lambda)}(\varphi-\pi/2)^2 + \mathcal{O}(\varphi-\pi/2)^3$$

where $\mathcal{O}(\varphi-\pi/2)^3$ is uniform in λ. The complex Morse lemma then shows that the local analytic diffeomorphism $\theta \mapsto \widetilde{\theta}$ given by $\widetilde{\theta} = \sqrt{f_1(\theta-\theta_c;\varphi)}e^{-i\pi/4}(\theta-\theta_c)$, $f_1(0;\pi/2) = 1+i\lambda$, is such that $\Phi(\theta,\varphi) = \Phi(\theta_c,\varphi) - i\widetilde{\theta}^2/2$. Then complex stationary phase shows that the contribution of a (fixed) neighborhood of θ_c to the integral in (c.16) is given at leading order, by $\big(\frac{2\pi}{rf_1(0;\varphi)}\big)^{1/2} e^{i\pi/4} e^{-ir\Phi(\theta_c,\varphi)}$. Outside this neighborhood, non stationary phase arguments show that the integral is exponentially smaller than $e^{-ir\Phi(\theta_c,\varphi)}$, as a function of r, and we eventually get, for φ in a (real) neighborhood $V_\pm$ of $\pm\pi/2$:

$$(c.18) \qquad \int_{-\pi}^{\pi} d\theta e^{-ir\Phi(\theta,\varphi)} = \big(\frac{2\pi}{rf_1(0;\varphi)}\big)^{1/2} e^{i\pi/4} e^{-ir\Phi(\theta_c,\varphi)}(1 + R_1(\varphi,\lambda,r)/r)$$

with $|R_1(\varphi,\lambda,r)| \leq$ Const.. By analyticity, we can keep track of the critical point θ_c for all $\varphi \in [-\pi,\pi]$, so formula (c.18) still makes sense when $\varphi \notin V_\pm$, but then $\operatorname{Im}\Phi(\theta_c,\varphi) < t^*$, and $\varphi \notin V_\pm$ doesn't contribute to the final result. Now we raise (c.16) to the power N and integrate over $r \geq N$, using (c.18) this yields (with a factor 2, accounting for the contribution $\varphi \in V_-$)

$$(c.19) \qquad \begin{aligned} \int_{|v|\geq r_1} dv\varphi_m(v)^N \sim 2(2\pi\widehat{\phi}(t^*))^{-N} e^{iN\pi/4}(2\pi)^{N/2} \int_{r\geq N} rdr(rf_1(0;\varphi))^{-N/2} \\ \times \int_{V_+} d\varphi e^{-iNr\Phi_1(\varphi)}(1+R_1(\varphi,\lambda,r)/r)^N \end{aligned}$$

where $\Phi_1(\varphi) = \Phi(\theta_c,\varphi) - |m|\sin\varphi$. Then (c.17) shows that Φ_1 has a non degenerate point at $\varphi = \pi/2$, and as before, the complex Morse lemma gives a local analytic diffeomorphism $\varphi \mapsto \widetilde{\varphi} = f_2(\varphi-\pi/2)(\varphi-\pi/2)$ with $\frac{d\widetilde{\varphi}}{d\varphi}(\pi/2) = f_2(0) = \big(\frac{\lambda}{1+i\lambda} + i|m|\big)^{1/2}$, such that $\Phi_1(\varphi) - \Phi_1(\pi/2) = \frac{1}{2}\widetilde{\varphi}^2$, $\Phi_1(\pi/2) = 1-|m|+i\lambda$. So by complex stationary phase, with Nr as a large parameter, we can evaluate the inner integral in (c.19) :

$$(c.20) \qquad \begin{aligned} &\int_{V_+} d\varphi e^{-iNr\Phi_1(\varphi)}(1+R_1(\varphi,\lambda,r)/r)^N = \\ &\quad e^{-iNr(1-|m|)}e^{Nt^*}\sqrt{2\pi/Nr}f_2(0)^{-1}(1+R_1(\pi/2,\lambda,r)/r)^N(1+R_2(\lambda,|m|,r)/Nr) \end{aligned}$$

with $R_2(\lambda,|m|,r) \leq$ Const.. At last, we estimate the resulting r-integral in (c.19). This time we compute simply an upper bound for the integrand. We have $(1+R_1(\pi/2,\lambda,r)/r)^N \leq$ Const. uniformly in N as $r \geq N$, and the same holds for $(1+R_2(\lambda,|m|,r)/Nr)$, so inserting (c.20) into (1.19), we get

$$(c.21) \qquad \int_{|v|\geq r_1} dv\varphi_m(v)^N \leq \text{Const.}\int_{r\geq N} rdr\Big(\frac{e^{t^*}}{\sqrt{2\pi}I_0(t^*)}|r+it^*|^{-1/2}\Big)^N\sqrt{2\pi/Nr}|f_2(0)|^{-1}$$

We have

$$\Big(\frac{e^{t^*}}{\sqrt{2\pi}I_0(t^*)}|r+it^*|^{-1/2}\Big)^N = \Big(\frac{e^{t^*}}{\sqrt{2\pi}I_0(t^*)}(N^2+t^{*2})^{-1/4}\Big)^N\Big(\frac{N^2+t^{*2}}{r^2+t^{*2}}\Big)^{N/4}$$

and since $\frac{e^{t^*}}{\sqrt{2\pi t^*}I_0(t^*)} \to 1^-$ as $t^* \to +\infty$, it is easy to see that

$$\limsup_{N\to\infty}\Big(\frac{e^{t^*}}{\sqrt{2\pi}I_0(t^*)}(N^2+t^{*2})^{-1/4}\Big)^N \leq \text{Const.}$$

uniformly in $t^* > 0$. Furthermore $|f_2(0)|^{-1} = (t^{*2}+r^2)^{1/4}\big(t^{*2}(1-|m|)^2+r^2|m|^2\big)^{-1/4}$, and since $t^* \sim 2^{-1}(1-|m|)^{-1}$ as $|m| \to 1$, we have $|f_2(0)|^{-1} \leq \text{Const.}(1-|m|)^{-1/2}$ uniformly as $r \geq N$. So the integral on the RHS of (c.21) is bounded by a constant times

$$(1-|m|)^{-1/2}\int_{r\geq N} rdr\sqrt{2\pi/Nr}\Big(\frac{N^2+t^{*2}}{r^2+t^{*2}}\Big)^{N/4}$$

and it is easy to show that there is $q' \geq 0$ such that $\int_{r \geq N} r dr \sqrt{2\pi/Nr} \big(\frac{N^2+t^{*2}}{r^2+t^{*2}}\big)^{N/4} \leq$ Const. $N^{q'}$ uniformly in $t^* > 0$. This completes the proof of the Lemmc. ♣

Next we have the following :

Proposition c.2: With the notations above, there is $C_1 > 0$ such that

$$(c.25) \qquad |\log \frac{d\nu^{(\Lambda)}}{dm} + \gamma^{-d} I(m, \Lambda)| \leq C_1 (L\gamma^{-\delta})^d \log \gamma^{-1} + \log \prod_{x \in \widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}} \big(1 - |m(x)|\big)^{-1/2}$$

Proof: Sum (c.13) over all the cubes $\widetilde{C}^{(n_\gamma)}$ contained in $\widetilde{\Lambda}$, which have same cardinal $N = \gamma^{(\delta-1)d}$, and multiply by N the resulting equality. The first term on the LHS

$$(c.26) \qquad \log \prod_{\widetilde{C}^{(n_\gamma)}} \frac{d\nu^{(n_\gamma)}}{dm}(m) = \int_{\Omega_0} \prod_{i \in \widetilde{\Lambda}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big) \delta\big(\pi^{(n_\gamma)} \sigma_{\widetilde{\Lambda}}(i) - m\big)$$

is the Radon-Nikodym density $\frac{d\nu^{(\Lambda)}}{dm}(m)$ of a probability distribution, namely the family of empirical averages $\pi^{(n_\gamma)}$ in $\widetilde{C}^{(n_\gamma)}$ considered as i.i.d. random variables. Here we need interpret m as a $\mathcal{Q}^{(n_\gamma)}$-measurable function on $\mathbf{R}^d$, and the RHS of (c.24) should actually read

$$\int_{\Omega_0} \prod_{x \in \widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}} \prod_{i \in \widetilde{C}_x^{(n_\gamma)}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big) \delta\big(\pi^{(n_\gamma)} \sigma_{\widetilde{\Lambda}}(i) - m(x)\big)$$

where somewhat incorrectly, the notation $\widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}$ reminds us we have tiled $\widetilde{\Lambda}$ by translates of $\widetilde{C}^{(n_\gamma)}$, and $\widetilde{C}_x^{(n_\gamma)}$ is the atom of the partition of $\widetilde{\Lambda}$ labelled by x. We have $|\widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}| = |\Lambda/C^{(n_\gamma)}| = (L\gamma^{-\delta})^d$. This identification will be made freely in the sequel. Summing up the second terms on the LHS of (c.13) over x will produce $\gamma^{-d} I(m, \Lambda)$, where $I(m, \Lambda) = \int_\Lambda dr I(m(r))$ defines a functional on the space of $\mathcal{Q}^{(n_\gamma)}$-measurable functions. While summing up the RHS of (c.13) over x, (c.25) follows easily if we make use of the inequality $\log(a+b) \leq \log(1+a) + \log b$ valid for any $a > 0, b \geq 1$. ♣

c) Free energy estimates.

Following [Pr], we replace now in the hamiltonian the spins $\sigma_\gamma(r)$ by the magnetization $m_\gamma(r)$ as in (c.1). We have :

Lemma c.3: With the notations above, for some $C_2 > 0$ we have :

$$(c.30)) \qquad |H(m_\gamma | m_\gamma^c) - H(\sigma_\gamma | \sigma_\gamma^c)| \leq C_2 L^d \gamma^\delta \|\nabla J\|_1$$

(here $\| \cdot \|_1$ denotes the L^1-norm on $\mathbf{R}^d$)

Proof: By the definition of $H(\sigma_\gamma|\sigma_\gamma^c)$ and (c.1) we have, denoting $C_\gamma(r)$ for $\widetilde{C}^{(n_\gamma)}(r)$

$$
\begin{aligned}
&H(m_\gamma|m_\gamma^c) - H(\sigma_\gamma|\sigma_\gamma^c) = \big(\frac{1}{2}\int_\Lambda dr_1 \int_\Lambda dr_2 + \int_\Lambda dr_1 \int_{\Lambda^c} dr_2\big)\langle \sigma_\gamma(r_1), \sigma_\gamma(r_2)\rangle \\
&\times \big[\gamma^{-2d\delta}\int_{C_\gamma(r_1)} dr \int_{C_\gamma(r_2)} dr' \big(J(r-r') - J(r_1-r_2)\big)\big]
\end{aligned}
$$

where we have used $r_1 \in C_\gamma(r)$ iff $r \in C_\gamma(r_1)$. We estimate $J(r-r') - J(r_1-r_2)$ by Taylor formula, noticing that $|(r-r') - (r_1-r_2)| \le \text{Const.}\,\gamma^\delta$, so the dr_2 integrals over Λ or Λ^c are bounded by a constant times $\gamma^\delta \|\nabla J\|_1$, then the resulting dr_1 integral over Λ gives an additional L^d factor, which proves the Lemma. ♣.

Introduce the free energy in Λ at inverse temperature β, inclusive of the interaction on Λ^c, as the functional on the space of $\mathcal{Q}^{(n_\gamma)}$-measurable functions $m(r)$

$$
(c.31)\qquad
\begin{aligned}
\mathcal{F}(m|m^c) &= \frac{1}{4}\int_\Lambda dr \int_\Lambda dr' J(r-r')|m(r) - m(r')|^2 \\
&+ \frac{1}{2}\int_\Lambda dr \int_{\Lambda^c} dr' J(r-r')|m(r) - m(r')|^2 + \int_\Lambda dr \big(f_\beta(m(r)) - f_\beta(m_\beta)\big)
\end{aligned}
$$

where we recall $f_\beta(m) = -\frac{1}{2}|m|^2 + \frac{1}{\beta}I(m)$. Let $\widehat{e}$ be any (fixed) unit vector in $\mathbf{R}^q$, and $\widehat{m}_\beta$ the constant function on Λ equal to $m_\beta\widehat{e}$, which we extend to be equal to m^c on Λ^c. The functionals $I(\cdot, \Lambda)$ and $H(\cdot|m^c)$ are related to $\mathcal{F}(\cdot|m^c)$ in a simple way :

$$
(c.32)\qquad \mathcal{F}(m|m^c) - \mathcal{F}(\widehat{m}_\beta|m^c) = (H(m|m^c) + \frac{1}{\beta}I(m,\Lambda)) - (H(\widehat{m}_\beta|m^c) + \frac{1}{\beta}I(\widehat{m}_\beta, \Lambda))
$$

Analogously to (c.8), (c.2) we introduce the canonical Gibbs measure conditioned by the external configuration $\sigma_{\widetilde{\Lambda}^c} = \sigma_\gamma^c$:

$$
(c.33)\qquad \mu_{\beta,\gamma,\Lambda}(d\sigma_\gamma; m|\sigma_\gamma^c) = \frac{1}{Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c)}\int_{\Omega_0} \exp\big[-\beta\gamma^{-d}H_\gamma(\sigma_\gamma|\sigma_\gamma^c)\big]\prod_{i\in\widetilde{\Lambda}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)\delta(\pi_\gamma\sigma_\gamma(i) - m)
$$

where the partition function $Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) = Z_{\beta,\gamma,\widetilde{\Lambda}}(\sigma_{\widetilde{\Lambda}^c})$ was defined in (c.9). We have made use of (c.5) to work on the rescaled lattice Λ, and set $\delta(\pi_\gamma\sigma_\gamma(i) - m) = \delta\big(\pi^{(n_\gamma)}\sigma_{\widetilde{\Lambda}}(i) - m\big)$. By definition of the image of Gibbs measure through the block-spin transformation, we have

$$
(c.34)\qquad \int_{|m|<1} dm \mu_{\beta,\gamma,\Lambda}(d\sigma_\gamma; m|\sigma_\gamma^c) = 1
$$

where dm is the normalized Lebesgue measure on the product space $\prod_{x\in\Lambda^*} B_q(0,1)$. Next we give a precise meaning to the approximation $\mu_{\beta,\gamma,\Lambda} \approx \exp[-\beta\gamma^{-d}\mathcal{F}(m|m^c)]$ stated in the

Introduction, by establishing the analogue of [AlBeCaPr,Lemma 3.2] in case of continuous symmetry, improving also [BuPi, Lemma 3.1] :

Theorem c.4: With the notations above, there are constants $C_1, C_2 > 0$ such that for any coarse-grained configuration m on Λ, we have :

$$\begin{aligned} &-g(m) - (L\gamma^{-1})^d\big(C_2\beta\gamma^\delta\|\nabla J\|_1 + C_1\gamma^{(1-\delta)d}\log\gamma^{-1}\big) \\ (c.35) \qquad &\leq \log \mu_{\beta,\gamma,\Lambda}(d\sigma_\gamma; m|\sigma_\gamma^c) + \beta\gamma^{-d}\mathcal{F}(m|m^c) \\ &\leq g(m) + \inf_{\widehat{e}\in\mathbf{S^1}} \mathcal{F}(\widehat{m}_\beta|m^c) + (L\gamma^{-1})^d\big(C_2\beta\gamma^\delta\|\nabla J\|_1 + C_1\gamma^{(1-\delta)d}\log\gamma^{-1}\big)\end{aligned}$$

where $g(m) = \log\prod_{x\in\widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}}\big(1-|m(x)|\big)^{-1/2}$.

Proof: First we look for a lower bound on $Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c)$. Using (c.30) we get :

$$\exp[-\beta\gamma^{-d}H(\sigma_\gamma|\sigma_\gamma^c)] \geq \exp[-\beta\gamma^{-d}H(\pi_\gamma\sigma_\gamma|\sigma_\gamma^c)]\exp[-C_2\beta\gamma^{-d}L^d\gamma^\delta\|\nabla J\|_1]$$

(and similarly for the upper bound). Multiply these relations by $\delta(\pi_\gamma\sigma_\gamma(i)-m)$, integrate with respect to $\prod_{i\in\widetilde{\Lambda}}\nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)$ over Ω_0 and use (c.25), we get

$$\begin{aligned} &\exp[-\beta\gamma^{-d}(H(m|m^c) + \frac{1}{\beta}I(m,\Lambda))]\exp -\psi_\gamma(m) \\ (c.36) \qquad &\leq \int_{\Omega_0}\prod_{i\in\widetilde{\Lambda}}\nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)\exp[-\beta\gamma^{-d}H(\sigma_\gamma|\sigma_\gamma^c)]\delta(\pi_\gamma\sigma_\gamma(i)-m) \\ &\leq \exp[-\beta\gamma^{-d}(H(m|m^c) + \frac{1}{\beta}I(m,\Lambda))]\exp\psi_\gamma(m)\end{aligned}$$

where

$$(c.37) \qquad \psi_\gamma(m) = (L\gamma^{-1})^d\big(C_2\beta\gamma^\delta\|\nabla J\|_1 + C_1\gamma^{(1-\delta)d}\log\gamma^{-1}\big) + g(m)$$

Now we estimate the contribution of a neighborhood of $\widehat{m}_\beta$ to the partition function. So let $0\leq\chi\leq 1$ be a smooth positive cutoff equal to 1 near $m=\widehat{m}_\beta$, multiply (c.36) by $\chi(m)$ and integrate over m with respect to the product measure $\prod_{x\in\widetilde{\Lambda}/\widetilde{C}^{(n_\gamma)}} dm(x)$, then make use of (c.9), (c.32) and (c.34), we get

$$\begin{aligned} Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) \geq &\exp[-\|\psi_\gamma\|_\chi]\exp[-\beta\gamma^{-d}(H(\widehat{m}_\beta|m^c) + \frac{1}{\beta}I(\widehat{m}_\beta,\Lambda))] \\ &\times\int dm\chi(m)\exp[-\beta\gamma^{-d}(\mathcal{F}(m|m^c) - \mathcal{F}(\widehat{m}_\beta|m^c)]\end{aligned}$$

where $\|\psi_\gamma\|_\chi = \sup_{m\in\mathrm{supp}\,\chi}\psi_\gamma(m) < \infty$. Choose supp χ so small that $|m-\widehat{m}_\beta| \leq \gamma^{(1-\delta)d}$ on supp χ. Using (c.31), the normalisation of J and Taylor expansion of f_β around $\widehat{m}_\beta$, we get $|\mathcal{F}(m|m^c) - \mathcal{F}(\widehat{m}_\beta|m^c)| \leq C_3L^d\gamma^{(1-\delta)d}$ on supp χ, so :

(c.38)

$$Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) \geq \exp[-\big(\|\psi_\gamma\|_\chi + C_3(L\gamma^{-1})^d\beta\gamma^{(1-\delta)d}\big)]\exp[-\beta\gamma^{-d}(H(\widehat{m}_\beta|m^c) + \frac{1}{\beta}I(\widehat{m}_\beta,\Lambda))]$$

Inserting this in (c.33) and (c.36) we find

$(c.39)$

$$\log \mu_{\beta,\gamma,\Lambda}(d\sigma_\gamma; m|\sigma_\gamma^c) \le -\beta\gamma^{-d}(\mathcal{F}(m|m^c)-\mathcal{F}(\widehat{m}_\beta|m^c))+\psi_\gamma(m)+\|\psi_\gamma\|_{\mathcal{X}}+C_3(L\gamma^{-1})^d\beta\gamma^{(1-\delta)d}$$

For the upper bound on $Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c)$, we use (c.34) and (c.36) to write

$$Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) = \int dm \int_{\Omega_0} \exp\big[-\beta\gamma^{-d} H_\gamma(\sigma_\gamma|\sigma_\gamma^c)\big] \prod_{i\in\widetilde{\Lambda}} \nu\big(d\sigma_{\widetilde{\Lambda}}(i)\big)\delta(\pi_\gamma\sigma_\gamma(i) - m)$$
$$\le \int dm \exp[-\beta\gamma^{-d}(H(m|m^c) + \frac{1}{\beta} I(m,\Lambda))] \exp\psi_\gamma(m)$$

By (c.32) and inequality $\mathcal{F}(m|m^c) \ge 0$, we have

$$Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) \le \exp[-\beta\gamma^{-d}\big(H(\widehat{m}_\beta|m^c) + \frac{1}{\beta}I(\widehat{m}_\beta,\Lambda)) - \mathcal{F}(\widehat{m}_\beta|m^c)\big)]$$
$$\times \int dm \exp\psi_\gamma(m)$$

se we are left to estimate $\int dm \exp\psi_\gamma(m)$, the integral running over the product space, and $\psi_\gamma(m)$ as in (c.37). Since $\int_0^1 (1-\rho)^{-1/2}\rho d\rho < \infty$ we find

$$Z_{\beta,\gamma,\Lambda}(\sigma_\gamma^c) \le \exp[-\beta\gamma^{-d}\big(H(\widehat{m}_\beta|m^c) + \frac{1}{\beta}I(\widehat{m}_\beta,\Lambda) - \mathcal{F}(\widehat{m}_\beta|m^c)\big)]$$
$$\times \exp[(L\gamma^{-1})^d\big(C_2\beta\gamma^\delta\|\nabla J\|_1 + C_1\gamma^{(1-\delta)d}\log\gamma^{-1} + C_4\gamma^{(1-\delta)d}\big)]$$

Inserting this and the first inequality (c.36) in (c.33), we find, absorbing the C_4-remainder term into the C_1-remainder term, and using

$$\begin{aligned} \log \mu_{\beta,\gamma,\Lambda}(d\sigma_\gamma; m|\sigma_\gamma^c) \ge& -g(m) - \beta\gamma^{-d}\mathcal{F}(m|m^c) \\ &- (L\gamma^{-1})^d\big(2C_2\beta\gamma^\delta\|\nabla J\|_1 + 2C_1\gamma^{(1-\delta)d}\log\gamma^{-1}\big) \end{aligned} \qquad (c.43)$$

Putting (c.43) together with (c.39) with new constants C_1, C_2 gives the Theorem. ♣

Of course,these estimates break down when $|m(x)|$ gets close to 1 for some $x \in \Lambda$, which reflects the fact that the entropy density $I(m)$ is singular near $|m| = 1$. It is shown in [BuPi,Theorem 2.2] that $\nu_N(\{|m| > 1-\rho\})$ decays exponentially fast as $N \to \infty$ when $\rho > 0$ is small enough.

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